

LAGRANGIAN FLOER THEORY IN SYMPLECTIC FIBRATIONS

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ABSTRACT. Consider a fibration of compact symplectic manifolds $F \rightarrow E \rightarrow B$ with a compatible symplectic form on E , and an induced fibration of Lagrangian submanifolds $L_F \rightarrow L \rightarrow L_B$. We develop a Leray-Serre type spectral sequence to compute the Floer cohomology of L in terms of the Floer complex of L_F and L_B . To solve the transversality and compactness problem, we use the classical approach in addition to the perturbation scheme recently developed by Cieliebak-Mohnke [7] and Charest-Woodward [5, 4]. As applications, we find Floer-non-trivial tori in complex flag manifolds and ruled surfaces.

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1. INTRODUCTION

In many subfields of topology, one will not go very far without encountering the idea of a *fiber bundle* or a *fibration*. In a certain sense, this is the topological version of a short exact sequence. A fibration provides a natural way of viewing a large space as two smaller ones which are twisted together, or a way of constructing one space from two.

To say something about the topology of a fibration, one typically uses some sort of long exact sequence, or more generally a spectral sequence. This idea was made popular by Leray, Serre, Grothendieck, and others

[30, 22, 23, 19, 32]. For example, to compute the de Rham cohomology of a fiber bundle $F \rightarrow E \rightarrow B$, where B has a good cover \mathfrak{U} , one can use a spectral sequence whose second page is $E_2^{p,q} = H^p(\mathfrak{U}, \mathcal{H}^q)$, the Čech cohomology of the cover, where \mathcal{H}^q is the presheaf $U \mapsto H^q(\pi^{-1}(U))$. The idea goes back to one of Leray [22, 23], where he develops his spectral sequence to compute sheaf cohomology groups.

We would like to develop a Leray-Serre type spectral sequence in the setting of *pseudo holomorphic curves* and *Lagrangian Floer theory*. Pseudo holomorphic curves were introduced circa 1985 by Gromov [17], and have become a powerful tool in symplectic topology. One of the original applications was in defining a *quantum cup product* on the usual cohomology, which allows interactions between cocycles (or their Poincaré duals) which do not "intersect" in the classical situation. Further dynamical applications were considered by Floer and others [11, 12].

In this paper, the invariant of interest is *Lagrangian intersection Floer cohomology* [11, 26]. This theory takes as input two transversely intersecting Lagrangians (or often, a single Lagrangian) and in nice cases provides an obstruction to *displacement* by a Hamiltonian isotopy.

Fukaya et. al. (culminating in [13]) have discovered an underlying algebraic and categorical structure in the information given from Lagrangian intersection theory, called the *Fukaya category* of a symplectic manifold. Through homological mirror symmetry, the derived Fukaya category is expected to be naturally isomorphic to the derived category of coherent sheaves in the mirror manifold. Therefore, it seems feasible to try to find some generators for the Fukaya category, or at least some Floer non-trivial Lagrangians.

Let us denote the Floer cohomology of a single Lagrangian as $HF(L)$. This object is constructed as follows: We pick a Morse-Smale function on L and form the usual Morse complex $C(L)$. The Floer differential then counts *quantized* Morse flows: isolated pseudo holomorphic disks $u : (D, \partial D) \rightarrow (M, L)$ with boundary markings that map to specified stable/unstable manifolds. Assuming that we have made the right assumptions (L is monotone or weakly unobstructed) and have perturbed the almost complex structure correctly, this differential is well defined

and squares to zero, which gives us a homology theory.

In this paper, we study Lagrangians of the form $L_F \rightarrow L \rightarrow L_B$ contained in a fibration of symplectic manifolds $F \rightarrow E \rightarrow B$. Our main result is a spectral sequence which computes $HF(L)$ from the information of the Morse-Floer chain complexes $CF(L_F)$ and $CF(L_B)$. Each Morse-Floer configuration has a well defined *energy*, i.e. the symplectic area under pullback, which generates a discrete subgroup of \mathbb{R} . The energy of the configurations under the projection $\pi : E \rightarrow B$ provides a convenient filtration to induce a spectral sequence.

Let us now outline the project in further detail: consider a fibration of compact symplectic manifolds $(F, \omega_F) \rightarrow (E, \omega) \rightarrow (B, \omega_B)$ where ω is compatible with the fiber bundle structure; for instance,

$$\begin{aligned}\omega &= a + K\pi^*\omega_B \\ i^*a &= \omega_F\end{aligned}$$

Let us assume that F is monotone, and that B is rational (that is, ω_B has a non-zero representative in $H^2(B, \mathbb{Q})$). If given a monotone Lagrangian $L_F \subset F$ and a rational Lagrangian $L_B \subset B$, some natural questions one can ask are

- (1) Can we produce a Lagrangian $L \subset E$ as fiber bundle $L_F \rightarrow L \rightarrow L_B$ given some assumptions on the topology of $F \rightarrow E|_{L_B} \rightarrow L_B$
- (2) Given a Lagrangian $L \subset E$ of the form $L_F \rightarrow L \rightarrow L_B$, what can we say about the Floer cohomology of L given that of L_F and L_B

In this paper, much of the work will culminate to a definitive answer for (2). We will provide an answer for (1) in some special cases.

The main theorem is the following:

Theorem 1. *Let $(F, \omega_F) \rightarrow (E, \omega) \rightarrow (B, \omega_B)$ be a fibration of symplectic manifolds with (F, ω_F) monotone, $\omega_B \in H^2(B, \mathbb{Q})$, and ω as above. Suppose we have a fibration of Lagrangians $L_F \rightarrow L \rightarrow L_B$, with L_F monotone, L_B rational, L Lagrangian with respect to ω , and a divisor $D = \pi^{-1}(D_B)$ for a stabilizing divisor D_B of large enough degree in the base. Choose a regular, coherent, stabilizing, convergent perturbation datum (\mathcal{P}_Γ) . Then there is a spectral sequence $E_s^{p,q}$ which converges to $HF^*(L, \Lambda[r, q])$ whose second page is the Floer cohomology of the family of L_F over L_B . The latter is computed by a spectral*

sequence with second page

$$(1) \quad \tilde{E}_2 = H^*(L_B, \mathcal{HF}(L_F, \Lambda_{\geq 0}[r])) \otimes gr(\mathcal{F}_q \Lambda_{\geq 0}[q])$$

where the coefficients come from the system which assigns the module $HF(L_{F_p}, \Lambda_{\geq 0}[r])$ to each critical fiber.

The proof involves the usual transversality and compactness results for the moduli space of J -holomorphic disks in order to make the Floer cohomology well defined. We use a system of domain dependent almost complex structures, as developed in Cieliebak-Mohnke [7] and Charest-Woodward [5, 4], to overcome the multiple cover problem in achieving transversality in the base manifold. In order to make use of domain dependent perturbation data on the space $W^{k,p}(D, E, L)$, one needs the domain to be stable (no non-trivial automorphisms), since when defining the moduli of pseudo-holomorphic curves one identifies domains up to reparameterization. To stabilize our J -holomorphic domain configurations, we use the idea of a stabilizing divisor [7] (the existence attributed to [8]) which is typically Poincaré dual to some large multiple of the symplectic class. By requiring additional marked points on our configurations to map to the divisor, we obtain stable domains and can therefore use a more refined perturbation system.

Proving the transversality and compactness results in the fibration setting $F \rightarrow E \rightarrow B$ requires us to balance the aforementioned technique for a rational (B, L_B) with the more classical results for a monotone (F, L_F) . The main transversality result requires the use of an *upper triangular* perturbation system (with respect to a symplectic connection $TF \oplus H$) to show that the linearized Cauchy-Riemann operator is surjective in the particular case that a disk is constant along the fibers. One can then apply the classic density argument from [25] which uses the regularity for the adjoint of the linearized CR operator. The fact we are using domain-dependent perturbation data for B allows us to choose a section of $T_J \mathcal{J}$ which is only non-zero in a small neighborhood of some point p in the domain, thus bypassing the multiple cover problem inherent in the base manifold. For surjectivity in the fiber, we use the decomposition result for monotone manifolds due to Lazzarini [21]. This removes the need to stabilize components which are horizontally constant, and allows us to use a single almost complex structure for each component that is contained in a fiber. Compactness in this situation is a similar combination of techniques from the rational and monotone cases: basically, we use the divisor in the base to rule out any unstable bubble components under the projection, and

the classical type of regularization/dimension count to rule out vertical bubbles. The net result is that the only possibility for an unusual configuration in the limit is the formation of a stable disk component which does not break over critical points and is non-constant in the horizontal direction. Due to the assumption that the minimal Maslov index of L_F is 2, we do get the usual disk bubble connected to a constant disk, which cancels in the differential due to the different orderings of the boundary markings.

In order to write down a spectral sequence, we use coefficients from $\Lambda_{\geq 0}[q, r]$, the Novikov ring with discrete powers of q and r , with q appearing as $q^{E(\pi \circ u)}$ in the differential, and r appearing as $r^{E(u) - E(\pi \circ u)}$. Filtering the complex $CF(L, \Lambda_{\geq 0}[q, r])$ with respect to q degree induces a spectral sequence similar to the one in [13] section 6.2. However, the result here is that the second page is the homology of the complex $CF(L, \Lambda_{\geq 0}[q, r])$ but with respect to the differential d^0 which counts configurations with no q degree.

A similar result, in the form of a Künneth theorem for Fukaya algebras of Lagrangians, appears in work due to Amorim [1]. As far as we can tell, the main difference from this work seems to be the result of a balancing act: Amorim describes the A_∞ algebra of a product Lagrangian, while we describe less of the algebraic structure ($HF(L)$) in a more topologically complicated setting.

In future versions of this paper, we will attempt to write down a formula relating the *potential* functions of the base, fiber, and total space. It is believed that this should not be too hard once one considers using coefficient from a Novikov ring in two variables. Conjecturally, the potential for the base should just be given by setting $r = 0$ in the potential for the total space.

The immediate product of these technical results is a Floer cohomology theory that accepts as input Lagrangian fibrations $L_F \rightarrow L \rightarrow L_B$. In particular, this extends the theory in the rational or monotone cases, and allows for some new applications. It is now possible to find some Floer-non-trivial tori in certain classes of *minimal models*, e.g. \mathbb{P}^1 bundles over a Riemann surface; we compute some lower dimensional examples at the end of the paper. The implication of this is further reaching than one would expect, due to a program of Gonzalez-Woodward [15,

33]. In their program, they use the minimal model program from algebraic geometry to produce Floer-non-trivial generators for the Fukaya category. The starting point is what some refer to as a *Mori* fibration, and at each stage of a *running* of the minimal model program, more generating Lagrangians are created, which then persist to the beginning of the running, i.e. the original space. Thus, finding Floer-non-trivial Lagrangians in a Mori fibration will (in nice cases) give Floer-non-trivial Lagrangians in the original space. Moreover, the end stage Mori fibration typically has Fano fiber. This motivates the following definition:

Definition 1. A *Symplectic Mori Fibration* is a fiber bundle of symplectic manifolds $(F, \omega_F) \rightarrow (E, \omega) \xrightarrow{\pi} (B, \omega_B)$, whose transition maps are symplectomorphisms of the fibers, (F, ω_F) is monotone, (B, ω_B) is rational, and $\omega = a + K\pi^*\omega_B$ for large K with $\iota^*a = \omega_F$.

The assumptions of *rational* and *monotone* are necessary to make the Floer theory work.

In addition to the Mori surfaces exemplified at the end of this paper, the following example of *full flags* has been a toy model for this project.

1.1. Example: Full Flags. We prove there there is a Floer non-trivial 3-torus T in the three dimensional complex flag manifold which fibers over the Clifford torus in \mathbb{P}^2 . As far as the author knows, this is has not been exposed in the literature (compare [14, 27]).

Consider the space of nested complex vector spaces $V_1 \subset V_2 \subset \mathbb{C}^3$. We can realize this as a symplectic fiber bundle $\mathbb{P}^1 \rightarrow \text{Flag}(\mathbb{C}^3) \rightarrow \mathbb{P}^2$, with the both the base and fiber monotone. The type of Lagrangian that we are looking for is of the form $L_F \rightarrow L \rightarrow L_B$, where L_B and L_F are the so-called Clifford tori in \mathbb{P}^n . More generally, L_F is any smooth, simple, closed curve which divides the symplectic area of \mathbb{P}^1 into halves. By the Riemann mapping theorem, the Floer cohomology of L_F is isomorphic to that of any equator. Such an L constructed this way *should* be non-displacable, and we describe the construction after some preliminaries.

Holomorphic (but not symplectic) trivializations for $\text{Flag}(\mathbb{C}^3)$ can be realized as follows. Start with a chain of subspaces $V_1 \subset V_2 \subset \mathbb{C}^3$ with $V_1 \in \mathbb{P}^2$ represented as $[z_0, z_1, z_2]$ with $z_0 \neq 0$. Using the reduced row echelon form, there is a unique point in $\mathbb{P}(V_2)$ with first coordinate

zero, $[0, w_1, w_2]$. On the open set U_0 of \mathbb{P}^2 , we get a trivialization

$$\begin{aligned}\Psi_0 : \text{Flag}(\mathbb{C}^3) &\rightarrow U_0 \times \mathbb{P}^1 \\ ([z_0, z_1, z_2], V_2) &\mapsto ([z_0, z_1, z_2], [w_1, w_2])\end{aligned}$$

If $z_1 \neq 0$, then the transition map $U_0 \times \mathbb{P}^1 \rightarrow U_1 \times \mathbb{P}^1$ is given by

$$g_{01}([w_1, w_2]) = \left[-\frac{z_0 w_1}{z_1}, w_2 - \frac{z_2 w_1}{z_1}\right]$$

which is a well defined element

$$\begin{bmatrix} \frac{-z_0}{z_1} & 0 \\ \frac{-z_2}{z_1} & 1 \end{bmatrix}$$

in $PGL(2)$. A similar transition matrix works for the other trivializations.

Unfortunately, the above *algebraic* viewpoint does not contain any sort of symplectic structure. There is a natural symplectic form that we could use given by viewing $\text{Flag}(\mathbb{C}^3)$ as a coadjoint orbit $SU(3)/T$ with

$$\omega_\xi(X, Y) = \xi([X, Y])$$

where X, Y are in $\mathfrak{su}(3)/\{\mathfrak{stab}(\xi)\}$ [31]. This is $SU(3)$ equivariant, and thus the action of $SU(3)$ gives symplectomorphisms of the fibers.

On the other hand, finding a fibered Lagrangian requires a careful argument based on results from Guillemin-Lerman-Sternberg [20]. In \mathbb{P}^n , there is a distinguished *Clifford torus*, denoted $\text{Cliff}(\mathbb{P}^n)$ of the form

$$[z_0, \dots, z_n] : \|z_i\| = \|z_j\| \forall i, j$$

which is also realized as the central moment fiber with regard to the action of T^n . It was demonstrated in [6] that this is a monotone, Floer-nontrivial Lagrangian. In \mathbb{P}^1 , this is merely an equator with respect to a Hamiltonian height function. The main idea is that we want to find a Lagrangian sub-bundle

$$\text{Cliff}(\mathbb{P}^1) \rightarrow L \rightarrow \text{Cliff}(\mathbb{P}^2)$$

for which we will be able to compute the Floer cohomology.

The relevant result that we will use gives a description of the moment map for a symplectic fibration over a Hamiltonian base manifold, which will trivialize the fibration above $\text{Cliff}(\mathbb{P}^2)$. Let $F \rightarrow E \rightarrow B$ be a symplectic fibration with a compact G -action for which the projection is equivariant. Denote ψ as the moment map for the action of G on B . Let Δ be an open set of the moment polytope for which the action is

free. Given these assumptions, the discussion in [20] section 4.6 leads to the following theorem:

Theorem 2. [20] *Over $U = \psi^{-1}(\Delta)$, there is a symplectic connection Γ such that the moment map for the action on $\pi^{-1}(E)$ with the weak coupling form $\omega_\Gamma + \pi^*\omega_U$ is $\psi \circ \pi$*

See chapter 4 of [20] for a proof.

In lieu of the ability to change the connection on an open set (see the G -equivariant versions of theorems 5 and 6), this new symplectic structure is not much different from (in fact, isotopic to) the weak coupling form associated the original fiber-wise structure.

We sketch the proof of this theorem, as well as how it ties into our example: The key component involves constructing a space E_W which is a symplectic fibration over the family of reduced spaces W , and one obtains a new symplectic connection (and associated weak coupling form) on $E|_U \rightarrow U$ by pulling back the connection from this new space. Moreover, the fibration $E_W \rightarrow W$ can be shown to induce a fibration of reduced spaces $(\psi \circ \pi)^{-1}(\alpha)/G \rightarrow \psi^{-1}(\alpha)/G$. In our situation, we take $G = T^2$, $\psi : \mathbb{P}^2 \rightarrow \mathfrak{t}^\vee$ to be the associated moment map, and α as the barycenter of the moment polytope for \mathbb{P}^2 . Thus, modified connection on $E|_U \rightarrow U$ is trivial over $\psi^{-1}(\alpha)$ due to the fact that it is induced from $(\psi \circ \pi)^{-1}(\alpha)/G \rightarrow \{point\}$. Thus, the fibration is symplectically trivial above $\text{Cliff}(\mathbb{P}^2)$ with the new connection.

We are now free to pick a Lagrangian in the form $L = L_F \times T^2 \subset \text{Flag}(\mathbb{C}^3)$ above the central moment fiber with L_F dividing the symplectic area of the fibers in half. Applying the aforementioned change-of-connection in reverse then gives us a Lagrangian in the original weak coupling form which fiber-wise resembles L_F .

Let us now pick a Morse-Smale function on $\text{Cliff}(\mathbb{P}^2)$, such as the sum of two height functions $h_1 + h_2$. In the case that the Lagrangian we pick is trivially $\text{Cliff}(\mathbb{P}^2) \times \text{Cliff}(\mathbb{P}^1)$, we can use the three-way sum of S^1 height functions $h_1 + h_2 + h_3$ as our Morse-Smale function. Alternatively, one can follow a standard recipe when the fibration is non-trivial: Choose a Morse-Smale function on each critical fiber $\pi^{-1}(x_i)$ and extend to the rest of the space using cutoff functions in local trivializations. Explicitly, let $\phi_i : \text{Cliff}(\mathbb{P}^2) \rightarrow \mathbb{R}$ be a cutoff function which is 1 in a neighborhood of x_i and 0 outside of some local trivialization $U_i \ni x_i$, with the U_i disjoint. Pick an identification of each critical fiber

$\pi^{-1}(x_i)$ with S^1 , a height function $g : S^1 \rightarrow \mathbb{R}$, and form

$$f(p) = h_1 \circ \pi(p) + h_2 \circ \pi(p) + \sum_{i=0}^3 \phi_i \circ \pi(p) g(\theta)$$

We will assume that we can perturb this function in a neighborhood near each critical point to make it Morse-Smale and not change the individual critical points.

Following [6], the maslov index 2 disks in the base with boundary in $\text{Cliff}(\mathbb{P}^2)$ are of the form

$$\phi_0(z) = [z, 1, 1]$$

We have similar formulas for ϕ_1 and ϕ_2 . This works analogously for the fiber $\text{Cliff}(\mathbb{P}^1) \subset \mathbb{P}^1$.

Following from Grauert's *h-principle* [16], which we discuss later, holomorphic disks with boundary in L are precisely products of disks of the above form, due to the fact that $(\pi \circ u)^* \text{Flag}(\mathbb{C}^3)$ is holomorphically $D \times \mathbb{P}^1$ for u holomorphic. To make for an even nicer situation, both the base and fiber are monotone, so the A_∞ algebra is already curvature free (this follows from the compactness argument in section 4.4).

We work over the power series ring $\Lambda = \mathbb{C}[[r^\eta, q^\rho]]$ with η resp. ρ as half of the energy corresponding to the minimal maslov index for the fiber resp. base Lagrangian. Let us order the critical points on the base T^2 by x_0, x_1, x_2, x_3 in order of *increasing* dimension of their *stable manifolds* $W^+(x_j)$. Then let x_j^i be a lift of the x_j such that the dimension of $W^+(x_j^i)$ restricted to the fiber is 0 resp. 1 for $i = 0$ resp. $i = 1$. Using the aforementioned classification of holomorphic disks and the most basic version of the index formula for configurations with root x_0

$$\iota(\Gamma, x_0, \dots, x_n) := \dim W_f^+(x_0) - \sum_{i=1}^n \dim W_f^+(x_i) + \sum_{i=1}^n I(u_i) + n - 2$$

we can compute the following for the Floer differential d :

$$\begin{aligned}
d(x_0^0) &= 0; \\
d(x_0^1) &= r^{2\eta}x_0^0 - r^{2\eta}x_0^0 = 0 \\
d(x_j^0) &= q^{2\rho}x_0^0, \quad j = 1, 2 \\
d(x_j^1) &= r^{2\eta}x_j^0 - r^{2\eta}x_j^0 + q^{2\rho}x_0^1 = q^{2\rho}x_0^1, \quad j = 1, 2 \\
d(x_3^0) &= q^{2\rho}x_1^0 - q^{2\rho}x_2^0 \\
d(x_3^1) &= q^{2\rho}x_1^1 - q^{2\rho}x_2^1 + r^{2\eta}q^{2\rho}x_0^0 - r^{2\eta}q^{2\rho}x_0^0 = q^{2\rho}x_1^1 - q^{2\rho}x_2^1
\end{aligned}$$

where the orientations have been chosen so that they agree with the same orientations given on the 1-dimensional part of the moduli space. Notice that there are cross terms $\pm r^{2\eta}q^{2\rho}x_0^0$ in the differential of x_3^1 which cancel with each other: These seem to show up as an indication that there *could* be more structure coming from the fibration. We get that

$$HF(L) \cong \Lambda\{x_0^i, x_2^i - x_1^i\} \Big/ q^{2\rho} \Lambda\{x_0^i, x_2^i - x_1^i\}$$

which shows this Lagrangian as non-displaceable.

In section 5, we give a construction for a fibered Floer-non-trivial Lagrangian in higher dimensional flag manifolds and use the spectral sequence to compute its Floer cohomology.

1.2. Outline. The paper is divided into five sections and an appendix. In section 2, we follow the literature to lay the necessary groundwork to discuss symplectic fiber bundles.

In section 3, we give a review of Floer theory for *rational* symplectic manifolds, as developed in [4, 7].

In section 4, we prove the transversality and compactness results in the fibration setting, and state the main theorem of this paper.

Section 5 is devoted to explicit examples of computations in the case of a ruled surface and a more general flag manifold.

The appendix is background taken from [4], and was included for future versions of this paper which will include statements about the *potential* function [13].

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2. SYMPLECTIC FIBRATIONS

We would like to unravel what we mean by the definition of a *symplectic Mori fibration* 1, and state some results pertaining to our situation. The idea is that we require the transition maps of our fiber bundle to be symplectomorphisms of the fibers. We then get a connection from the section $p \mapsto \omega_p$ by taking the symplectic complement TF^\perp , which allows us to parallel transport. Then, if B is also symplectic and $\iota^*a = \omega_F$, then the form $\omega_F + K\pi^*\omega_B$ is non-degenerate for large K . The main obstruction in this setup is finding a closed form a on E such that $\iota^*a = \omega_F$. Let's elaborate.

Following [24] chapter 6, we start with fiber bundle with connected total space E with a compact symplectic base (B, ω_B) and fiber (F, ω_F) . A *symplectic fibration* is such a space E where the transition maps are symplectomorphisms of the fibers. Then, we have a canonical symplectic form on each fiber ω_{F_p} given by the local trivializations, or the injections $\iota : F \rightarrow F_p$. Assume that there is a class $[a] \in H^2(E)$ such that $\iota^*[a] = [\omega_{F_p}]$. Then for large K , a theorem of Thurston (Theorem 6.3 in [24]) tells us that there is a symplectic form ω_K on E which represents the class $[a + K\pi^*\omega_B]$ and is compatible with the fibration structure.

Theorem 3 (Thurston [24]). *Let $(F, \omega_F) \rightarrow E \rightarrow (B, \omega_B)$ be a compact symplectic fibration with connected base. Let ω_{F_p} be the canonical symplectic form on the fiber F_p and suppose that there is a class $a \in H^2(M)$ such that*

$$\iota_b^*a = [\omega_{F_b}]$$

for some (and hence every) $b \in B$. Then, for every sufficiently large real number $K > 0$, there exists a symplectic form $\omega_K \in \wedge^2(T^\vee E)$ which makes each fiber into a symplectic submanifold and represents the class $a + K[\pi^\omega_B]$*

Generally, for the existence of the class a , one has to assume that F is simply connected or a surface of genus $g \neq 1$. In the case that F is a surface, we have the following lemma from [24]:

Lemma 1. *Let $(F, \omega_F) \rightarrow E \rightarrow (B, \omega_B)$ be a compact symplectic fibration such that the first Chern class $c_1(TF) = \lambda[\omega_F]$ for $\lambda \neq 0$. Then the class $\lambda^{-1}c_1(TM)$ pulls back to $[\omega_F]$*

One then applies Thurston's theorem to get a form τ on E which represents $\lambda^{-1}c_1(TM)$. Thus, if F is a Riemann surface but not a torus,

then E has compatible structure.

Let us write the form as $\tau_a + K\pi^*\omega_B$. Given that F_p is symplectic for τ_a , we get a well defined connection by taking the symplectic complement of TF , denoted $H = TF^{\perp\tau}$. We will call a connection arising in this way a *symplectic connection*, or equivalently a connection whose parallel transport maps are symplectomorphisms on the fibers. While there may be many (closed) such τ that define the same connection H , Guillemin-Lerman-Sternberg [20] and McDuff-Salamon [24] give a construction which uses the Hamiltonian action of parallel transport.

Theorem 4. [20, 24] *Let H be a symplectic connection on a fibration $F \rightarrow E \rightarrow B$ with $\dim F = n$. The following are equivalent:*

- (1) *The holonomy around any contractible loop in B is Hamiltonian.*
- (2) *There is a unique closed connection form ω_H on E with $i^*\omega_H = \omega_F$ and*

$$\int_F \omega_H^{(n+2)/2} = 0$$

where \int_F is the map from TB which lifts $v_1 \wedge v_2$ and integrates $\iota_{v_1 \wedge v_2} \omega_H^{2n+2}$ over the fiber.

The idea is that ω_H is already determined on vertical and vertical components, so it remains to describe it on horizontal components. This is done assigning the value of the zero-average Hamiltonian corresponding to $[v_1^\sharp, v_2^\sharp] - [v_1, v_2]^\sharp$, where the v_i^\sharp are horizontal lifts of base vectors v_i .

One might then ask: if we have two connection forms ω_{H_1} and ω_{H_2} , how are the symplectic forms $\omega_{H_1} + K\pi^*\omega_B$ and $\omega_{H_2} + K\pi^*\omega_B$ related. We have the following result.

Theorem 5. [20] *For two symplectic connections H_i , $i = 1, 2$, the corresponding forms $\omega_{H_i} + K\pi^*\omega_B$ are isotopic for large enough K .*

The hard part is actually finding a Lagrangian in the form $L_F \rightarrow L \rightarrow L_B$. If we can find such an L , it is not guaranteed to be Lagrangian due to small contributions from the horizontal part of ω_H . However, it seems feasible that we could alter the connection in a neighborhood of L to make it Lagrangian. Precisely, we have

Theorem 6. [20] *Let $U \subset B$ be an open set whose closure is compact and H' a symplectic connection for $\pi^{-1}(U)$. Then there is a connection H on E such that $H = H'$ over U .*

In light of theorem 5, nothing is lost if we modify the connection on our candidate Lagrangian and then extend it using theorem 6.

Methods to construct a submanifold $L \subset E$ of the form $L_F \rightarrow L \rightarrow L_B$ seem to be dependent on the situation. In the case when the ambient base manifold is dimension 2, we do not need to worry about horizontal contributions to ω_H and the obstruction is purely topological. In particular, we detail some examples of ruled complex surfaces in a later section of this paper.

3. FLOER THEORY FOR RATIONAL SYMPLECTIC MANIFOLDS

3.1. Moduli space of treed stable disks. In this section we record the results of Charest-Woodward [4], based on the results of Cieliebak-Mohnke [7]. They prove transversality and compactness for *rational*, non-fibered symplectic manifolds and Lagrangians [5, 4]. This section is included for completeness and will be adapted our use in later sections.

A fundamental problem in defining and Floer theory lies in making the right choices of perturbation data to resolve the problems of transversality and compactness. There are a number of popular methods, including the polyfolds approach and the method of Kuranishi structures. The author chose to use a more geometric approach developed in [7, 5, 4]. The main idea is to use the existence of a symplectic almost complex divisor which represents the Poincaré dual of (a large multiple) of the symplectic form [Donaldson] in order to stabilize domains and allow the use of domain dependent almost complex structures. We consider Morse-Floer trees that are stabilized by extra marked points that map to the divisor. We then show that we can choose an appropriate system of perturbation data that regularizes any reasonable configuration, including those with sphere or disk "bubbles". This regularization of bubble configurations allows us to then proof appropriate compactness results (which, in turn, rules out sphere bubbling).

A *tree* is a planar graph $\Gamma = (\text{Edge}(\Gamma), \text{Vert}(\Gamma))$ with no cycles which can be decomposed as follows:

- (1) For nonempty $\text{Vert}(\Gamma)$, $\text{Edge}(\Gamma)$ consists of
 - (a) *finite edges* $\text{Edge}_{<\infty}(\Gamma)$ connecting two vertices
 - (b) *semi-infinite edges* Edge_{∞} with a single endpoint, or

- (2) if $\text{Vert}(\Gamma)$ is empty, then Γ has one *infinite edge* and let Edge_∞ denote its two ends.

From $\text{Edge}_\infty(\Gamma)$ we can distinguish one open endpoint as the *root* or the tree, and the other semi-infinite edges being referred to as the *leaves*. A *metric tree* is a tree with an assignment of length to each finite edge, denoted $l: \text{Edge}_{<\infty}(\Gamma) \rightarrow [0, \infty]$. If a finite edge has infinite length, we call that edge broken, and thus we have a *broken metric tree*. We think of this as two metric trees, where the first has a leaf with extremal point ∞_1 , which is glued to the extremal point ∞_2 of the root of the second. Finally, a broken metric tree is *stable* if the valence of each vertex is at least 3.

A *nodal n -marked disk* is a collection of holomorphic disks which are identified at boundary nodes in a way that the total space is contractible. We equip markings $\{z_1, \dots, z_n\}$ which are labeled in accordance with a cyclic order around the boundary. A nodal marked disk is *stable* if each disk component has at least 3 nodes or markings.

The above is a realization of Stasheff's associahedron as a moduli space of geometric objects. In particular, isomorphism classes of stable nodal n -marked disks and stable trees with n semi-infinite edges form a compact cell complex, the later being a quotient of the former. It follows that the closure of broken stable metric trees with n semi-infinite edges form a cell complex.

For this particular scheme, we will also allow interior markings and holomorphic spheres in the definition of nodal disks. The holomorphic spheres will have nodes which are attached to interior points of disks and other spheres.

Definition 2. [4] A *treed disk* C is a triple (T, D, o) consisting of

- (1) A broken metric tree $T = (\Gamma, l)$
- (2) A collection $(S_v, \underline{x}_v, \underline{z}_v)_{v \in \text{Vert}(\Gamma)}$ of marked nodal disks for each vertex v of T , with the number of boundary markings \underline{x}_v equal to the valence of v
- (3) An ordering o of the set of interior markings $\cup_v \underline{z}_v \in \text{int}(D)$, so that we may denote the interior markings z_1, \dots, z_m .

We will be studying J -holomorphic maps from a geometric realization of C , given by replacing the vertices with their corresponding nodal disks by attaching the boundary markings \underline{x}_v to the appropriate edges at v . A treed disk is *stable* iff

- (1) The tree T is stable, i.e. the valence of each vertex is at least three
- (2) Each nodal disk S_v is stable. That is, each S_v contains at least three special points, or one interior marking and one boundary point

An equivalence of broken metric trees identifies any finite edge with infinite length with two semi-infinite edges, which are identified at their respective ∞_1, ∞_2 . An equivalence of treed disks is an equivalence of the underlying metric trees and an isomorphism of each marked nodal disk which preserves the ordering of the interior markings.

The combinatorial type of a treed disk $C = (T, D, o)$ includes the type of tree Γ obtained by gluing (into T) the tree of each nodal marked disk $\Gamma(D_v)$ at the corresponding markings (semi-infinite edges of $\Gamma(D_v)$), as well as:

- (1) the set of edges $\text{Edge}_{<\infty}(\Gamma)$ of length 0 or ∞ , and
- (2) the set of $\text{Edge}_{<\infty}(\Gamma)$ with finite non-zero length.

The vertices partition into the set

$$\text{Vert}(\Gamma) = \text{Vert}_d(\Gamma) \sqcup \text{Vert}_s(\Gamma)$$

and the edges:

$$\begin{aligned} \text{Edge}(\Gamma) = & \text{Edge}_{<\infty,s}(\Gamma) \sqcup \text{Edge}_{<\infty,d}(\Gamma) \sqcup \text{Edge}_{\infty,s} \\ & \sqcup \text{Edge}_{<\infty}^0(\Gamma) \sqcup \text{Edge}_{<\infty}^\infty(\Gamma) \sqcup \text{Edge}_{<\infty}^{(0,\infty)}(\Gamma) \sqcup \text{Edge}_\infty(\Gamma) \end{aligned}$$

which are the spherical nodes, boundary nodes, interior markings, finite edges with zero, infinite, and finite non-zero length, as well as semi-infinite edges.

We encode this data into a moduli space of stable treed disks $\mathfrak{M}^{n,m}$, where n is the number of semi-infinite edges and m the number of interior markings.

For a given stable combinatorial type Γ , let \mathfrak{M}_Γ be the strata of treed disks of type Γ . We have a universal treed disk of type $\mathcal{U}_\Gamma \rightarrow \mathfrak{M}_\Gamma$ which consists of points (C_m, m) , where m is of type Γ and C_m is its geometric realization. We can view a universal treed disk as a union of two sets: $S_\Gamma \cup T_\Gamma$. The former being the two dimensional part of each fiber, and the later being the one dimensional part. $S_\Gamma \cap T_\Gamma$ is the set of nodes and boundary markings. Given a treed disk C , we can identify nearby disks with C using a local trivialization. This gives us a map

for each chart

$$(2) \quad \mathfrak{M}_\Gamma^i \rightarrow \mathcal{J}(C)$$

where $\mathcal{J}(C)$ are holomorphic structures on the surface part of C .

Definition 3. [4](Behrend-Manin morphisms of graphs) A *morphism* of graphs $\Upsilon : \Gamma \rightarrow \Gamma'$ is a surjective morphism on the set of vertices obtained by combining the following elementary morphisms:

- (a) (Cutting edges) Υ *cuts an edge* $e \in \text{Edge}_{<\infty}(\Gamma)$ with infinite length resp. an edge $e \in \text{Edge}_{\infty,s}(\Gamma)$ (spherical node) if the map on vertices is a bijection, but

$$\text{Edge}(\Gamma') \cong \text{Edge}(\Gamma) - \{e\} + \{e_+, e_-\}$$

where $e_\pm \in \text{Edge}_\infty(\Gamma')$ are attached to the vertices contained in e . We view Γ' as two disconnected graphs Γ_+, Γ_- .

- (b) (Collapsing edges) Υ *collapses an edge* if the map on vertices $\text{Vert}(\Upsilon) : \text{Vert}(\Gamma) \rightarrow \text{Vert}(\Gamma')$ is a bijection except for two vertices in $\text{Vert}(\Gamma)$ which are joined by an edge in $\text{Edge}_{<\infty}^0(\Gamma)$.

$$\text{Edge}(\Gamma) \cong \text{Edge}(\Gamma') - \{e\}$$

- (c) (Making an edge length finite or non-zero) Υ *makes an edge finite or non-zero* if Γ is the same graph as Γ' and the lengths of the edges $l(e)$ for $e \in \text{Edge}_{<\infty}(\Gamma')$ are the same except for a single edge for which $l(e) = \infty$ resp. 0 and the length $l'(e)$ in Γ' is in $(0, \infty)$.
- (d) (Forgetting tails) Υ forgets a semi-infinite edge and collapses edge to make the resulting combinatorial type stable. The ordering on $\text{Edge}_{\infty,s}(\Gamma)$ naturally defines one on $\text{Edge}_{\infty,s}(\Gamma')$.

Each of the above operations on graphs corresponds to a map of moduli spaces of stable marked treed disks.

Definition 4. [4] (Morphisms of moduli spaces)

- (a) (Cutting edges) Suppose that Γ' is obtained from Γ by cutting an edge of infinite length. There are diffeomorphisms $\overline{\mathfrak{M}}_\Gamma \rightarrow \overline{\mathfrak{M}}_{\Gamma'}$ obtained by identifying the two endpoints corresponding to the cut edge and choosing the ordering of the interior markings to be that of Γ .
- (b) (Collapsing edges) Suppose that Γ' is obtained from Γ by collapsing an edge. There is an embedding $\overline{\mathfrak{M}}_\Gamma \rightarrow \overline{\mathfrak{M}}_{\Gamma'}$ whose image is a 1-codimensional corner or a 2-codimensional submanifold of $\overline{\mathfrak{M}}_{\Gamma'}$.

- (c) (Making an edge finite or non-zero) If Γ' is obtained from Γ by making an edge finite resp. non-zero, then $\overline{\mathfrak{M}}_\Gamma$ embeds in $\overline{\mathfrak{M}}_{\Gamma'}$ as the 1-codimensional corner where e reaches infinite resp. zero length, with trivial normal bundle.
- (d) (Forgetting tails) Suppose that Γ' is obtained from Γ by forgetting i -th tail, either in $\text{Edge}_{\infty,s}(\Gamma)$ or $\text{Edge}_\infty(\Gamma)$. Forgetting the i -th marking and collapsing the unstable components and their distance to the stable components (if any) defines a map $\overline{\mathfrak{M}}_\Gamma \rightarrow \overline{\mathfrak{M}}_{\Gamma'}$.

It is worthwhile to note that all of these maps extend to smooth maps of the corresponding universal treed disks. If Γ is disconnected, say the disjoint union of Γ_1 and Γ_2 , then the universal disk is the disjoint union of the pullbacks of the universal disks $\overline{\mathcal{U}}_{\Gamma_1}$ and $\overline{\mathcal{U}}_{\Gamma_2}$.

Orientations can be put on the space of treed disks as follows:

- (a) (For a single disk) For $m \geq 1$, we can identify any point in the open stratum of $\overline{\mathfrak{M}}_{n,m}$ with the half space $\mathbb{H} \subset \mathbb{C}$. To be consistent, say we map the root x_0 to ∞ , an interior marking z_1 to i and the boundary markings x_i to an $n-1$ -tuple of $\mathbb{R} \subset \mathbb{C}$. We then use standard orientations on these spaces.
If $m = 0$, send x_0 to ∞ , x_1 to 0, x_2 to 1, and the remaining boundary markings to an ordered tuple of $(1, \infty) \subset \mathbb{R} \subset \mathbb{C}$.
- (b) (Treed disks with multiple disk components) Given a treed disk in $\overline{\mathcal{U}}_{m,n}$ with an edge of zero length, we can realize it as being in the closure of a higher dimensional stratum by identifying the edge with a node. To obtain an edge of finite non-zero length, we use part (c) from the definition above. That is, the 1-codimensional corner where we have an edge of zero length is also realized as the boundary of the higher dimensional stratum where that edge has finite and non-zero length. Choose orientations on the top dimensional strata that induce the opposite orientations on the aforementioned 1-codimensional corners.

3.2. Treed holomorphic disks. Now that we have the notion of a treed disk, we can begin constructing the moduli of Floer trajectories. The vertices will represent domains for J -holomorphic maps while the edge parts will represent flow lines for a domain dependent Morse function.

Fix a metric G on L which extends to a metric on X for which L is totally geodesic. Pick a Morse-Smale function F on L which has a

unique maximum x_M . The gradient flow equation is the following:

$$(3) \quad \frac{d\phi_p(t)}{dt} = X_{\phi(t)}$$

where X_p is the gradient vector field of F with respect to g . If the critical point set is $\mathcal{I}(L)$, then for $x \in \mathcal{I}(L)$ denote the stable and unstable manifolds of x as

$$(4) \quad W_x^\pm(F)$$

respectively. The requirement that F is Morse-Smale guarantees that all of these submanifolds intersect transversely, and thus have smooth intersections. The index $I(x)$ is defined to be the dimension of W_x^+ . An almost complex structure for a X is a fiber-preserving linear map $J : TX \rightarrow TX$ such that $J^2 = -I$. J is tamed with respect to ω if $\omega(\cdot, J\cdot)$ is positive definite. Let $\mathcal{J}_\tau(X)$ denote the space of tamed almost complex structures.

The transversality scheme will involve Morse functions and almost complex structures which depend on the domain. However, we will need to fix sets in the domain on which the perturbation will be non-constant. Let $\overline{\mathcal{S}}_\Gamma \subset \overline{\mathcal{U}}_\Gamma$ be the two-dimensional part of the universal treed disk, and $\overline{\mathcal{T}}_\Gamma \subset \overline{\mathcal{U}}_\Gamma$ be the tree part of the universal treed disk. Fix a compact set

$$\overline{\mathcal{S}}_\Gamma^o \subset \overline{\mathcal{S}}_\Gamma$$

not containing the boundary, nodes, or interior markings, but having non-trivial intersection with every fiber of the universal disk \mathcal{U}_Γ . Also fix a compact set

$$\overline{\mathcal{T}}_\Gamma^o \subset \overline{\mathcal{T}}_\Gamma$$

having non-trivial intersection with each universal fiber. Thus, the compliments

$$\begin{aligned} \overline{\mathcal{S}}_\Gamma - \overline{\mathcal{S}}_\Gamma^o \\ \overline{\mathcal{T}}_\Gamma - \overline{\mathcal{T}}_\Gamma^o \end{aligned}$$

are neighborhoods of the boundary, interior markings, and nodes resp. neighborhoods of ∞ in each fiber of the universal disk.

Definition 5. [4]

- (a) (Domain-dependent Morse functions) Let (F, g) be a Morse-Smale pair, and $l > 0$ an integer. A *domain dependent perturbation* for F of class C^l is a C^l map

$$F_\Gamma : \overline{\mathcal{T}}_\Gamma \times L \rightarrow \mathbb{R}$$

equal to F on $\overline{\mathcal{T}}_\Gamma - \overline{\mathcal{T}}_\Gamma^o$.

- (b) (Domain-dependent almost complex structure) Let $J \in \mathcal{J}_\tau(X)$ an $l > 0$ an integer. A *domain-dependent almost perturbation for J* of class C^l for combinatorial type Γ is a C^l class map

$$J_\Gamma : \overline{\mathcal{S}}_\Gamma \rightarrow \mathcal{J}_\tau(X)$$

which is equal to J on $\overline{\mathcal{S}}_\Gamma - \overline{\mathcal{S}}_\Gamma^o$.

Thus, for a compact symplectic manifold (X, ω) we will use the following type of perturbation:

Definition 6. [4](Perturbation Data) A *perturbation datum* for combinatorial type Γ of class C^l is a pair $P_\Gamma = (F_\Gamma, J_\Gamma)$ consisting of a domain-dependent Morse function F_Γ and a domain-dependent almost complex structure J_Γ of class C^l .

We would like to choose perturbation datum which is compatible with operations on treed disks.

Definition 7. [4]

- (a) (Cutting edges) Suppose that Γ is a combinatorial type and Γ' is obtained by cutting an edge of infinite length. A perturbation datum of Γ' gives rise to a perturbation datum for Γ by pushing forward $P'_{\Gamma'}$ under the map $\overline{\mathcal{U}}'_{\Gamma'} \rightarrow \overline{\mathcal{U}}_\Gamma$
- (b) (Collapsing edges/making an edge finite or non-zero) Suppose that Γ' is obtained from Γ by collapsing an edge or making an edge finite or non-zero. Any perturbation datum $P'_{\Gamma'}$ for Γ' induces a datum for Γ by pullback of $P'_{\Gamma'}$ under $\overline{\mathcal{U}}'_{\Gamma'} \rightarrow \overline{\mathcal{U}}_\Gamma$.
- (c) (Forgetting tails) Suppose that Γ' is a combinatorial type of stable treed disk obtained from Γ by forgetting a marking. In the case there is a map of universal disks $\overline{\mathcal{U}}_{\Gamma'} \rightarrow \overline{\mathcal{U}}_\Gamma$ given by forgetting the marking and stabilizing. Any perturbation datum $P'_{\Gamma'}$ induces a datum P_Γ by pullback of $P'_{\Gamma'}$

Thus, it makes sense to define a perturbation datum which is compatible with the morphisms on graphs and moduli spaces. We will call this property *coherence*:

Definition 8. [4] A collection of perturbation data $\underline{P} = (P_\Gamma)$ is *coherent* if it is compatible with the morphisms of moduli spaces of different types in the sense that

- (a) (Cutting edges axiom) If Γ is obtained from Γ' by cutting an edge of infinite length, then $P_{\Gamma'}$ is the pushforward of P_Γ .
- (b) (Collapsing edges/making an edge finite or non-zero axiom) If Γ is obtained from Γ' by collapsing an edge or making an edge finite or non-zero, then $P_{\Gamma'}$ is the pullback P_Γ .

- (c) (Product axiom) If Γ is the union of types Γ_1, Γ_2 obtained from cutting an edge of Γ' , then P_Γ is obtained from P_{Γ_1} and P_{Γ_2} as follows: Let $\pi_k : \overline{\mathfrak{M}}_\Gamma \cong \overline{\mathfrak{M}}_{\Gamma_1} \times \overline{\mathfrak{M}}_{\Gamma_2} \rightarrow \overline{\mathfrak{M}}_{\Gamma_k}$ denote the projection onto the k th factor, so that $\overline{\mathcal{U}}_\Gamma$ is the unions of $\pi_1^* \overline{\mathcal{U}}_{\Gamma_1}$ and $\pi_2^* \overline{\mathcal{U}}_{\Gamma_2}$. Then we require that P_Γ is equal to the pullback of P_{Γ_k} on $\pi_k^* \overline{\mathcal{U}}_{\Gamma_k}$

Definition 9. [4] Given perturbation datum P_Γ , a *holomorphic treed disk* in X with boundary in L consists of a treed disk $C = S \cup T$ and a continuous map $u : C \rightarrow X$ such that

- (a) (Boundary condition) $u(\partial S \cup T) \subset L$.
(b) (Surface equation) On the surface part of S of C the map u is J -holomorphic for the given domain-dependent almost complex structure: if j denotes the complex structure on S , then

$$J_{\Gamma, u(z), z} du|_S = du|_S j.$$

- (c) (Tree equation) On the tree part $T \subset C$ the map u is a collection of gradient trajectories:

$$\frac{d}{ds} u|_\Gamma = -\text{grad}_{F_{\Gamma, (s, u(s))}}(u|_T)$$

where s is a local coordinate with unit speed, so that for each edge $e \in \text{Edge}_{<\infty}(\Gamma)$ the length of the trajectory, given by the length of $u|_{e \in T}$, is equal to $l(e)$.

A holomorphic treed disk $u : C \rightarrow X$ is *stable* iff

- (a) Each disk on which u is constant contains at least three special points or at least one interior special point and one other special point.
(b) Each sphere on which u is constant contained at least three special points.

We denote the moduli space of isomorphism classes of connected treed holomorphic disks with n leaves and m interior markings by $\mathcal{M}_{n,m}(L)$. For a connected combinatorial type Γ , $\mathcal{M}_\Gamma(L)$ denotes the subset of type Γ . $\mathcal{I}(L)$ is the set of critical points of F .

For a tuple of critical points $\underline{x} = (x_0, \dots, x_n)$ let $\mathcal{M}_\Gamma(L, \underline{x}) \subset \mathcal{M}_\Gamma(L)$ denote the subset of isomorphism classes of holomorphic treed disks u that have limits $\lim_{s \rightarrow \infty} u(\phi_{e_i}(s)) = x_i$ for $i \neq 0$ and $\lim_{s \rightarrow -\infty} u(\phi_{e_0}(s)) = x_0$.

The expected dimension of the moduli space is as follows:

$$i(\Gamma, \underline{x}) := I(x_0) - \sum_{i=1}^n I(x_i) + \sum_{i=1}^k I(u_i) + n - 2 - |\text{Edge}_{<\infty}^0(\Gamma)| \\ - |\text{Edge}_{\infty}(\Gamma) - (n+1)|/2 - 2|\text{Edge}_{<\infty,s}(\Gamma)| - \sum_{e \in \text{Edge}_{\infty,s}} m(e) - \sum_{e \in \text{Edge}_{<\infty,s}} m(e).$$

3.3. Transversality. In order to achieve transversality for the moduli space of stable treed J -holomorphic curves, we need to restrict to a slightly smaller class of symplectic manifolds and Lagrangian submanifolds:

Definition 10. [4] (Rationality)

- (a) A symplectic manifold (X, ω) is *rational* if the class $[\omega] \in H^2(X, \mathbb{R})$ is in the image of $H^2(X, \mathbb{Q})$; equivalently, if there is a *linearization* of X : a line bundle $\tilde{X} \rightarrow X$ with a connection whose curvature is $(2\pi k/i)\omega$ for $k \in \mathbb{Z}$.
- (b) Let $h_2 : \pi_2(X, L) \rightarrow H_2(X, L)$ be the relative Hurewicz map. Let $[\omega]^\vee : H_2(X, \mathbb{R}) \rightarrow \mathbb{R}$ be the map given by pairing with ω . A Lagrangian $L \subset X$ is *rational* if $[\omega]^\vee \circ h_2(\pi_2(X, L)) = \mathbb{Z} \cdot e$ for some $e > 0$.

We need the existence of a *stabilizing divisor* to kill any automorphisms of the domain so that our perturbation data descends to the quotient. The rationality assumptions allow the existence of such:

Definition 11. [4] (Stabilizing Divisors)

- (a) A divisor in X is a closed codimension two symplectic submanifold $D \subset X$. An almost complex structure $J : TX \rightarrow TX$ is adapted to a divisor D if D is an almost complex submanifold of (X, J) .
- (b) A divisor $D \subset X$ is *stabilizing* for a Lagrangian submanifold L if
 - (1) $D \subset X - L$, and
 - (2) There exists an almost-complex structure $J_D \in (\mathcal{J}, \omega)$ adapted to D such that any J_D holomorphic disk $u : (C, \partial C) \rightarrow (X, L)$ with $\omega([u]) > 0$ intersects D in at least one point.

We get the following theorem (from [5, 4, 7]) as an application of various techniques:

Theorem 7. *There exists a divisor $D \subset X$ that is stabilizing for L . Moreover, if L is rational then there exists a divisor $D \subset X$ that is stabilizing for L .*

We will need conditions on the interaction between the treed disks and the divisor:

Definition 12. [4] (Adapted stable treed disks) Let (X, L) be a symplectic manifold with Lagrangian L and a codimension two submanifold D disjoint from L . A nodal treed disk $u : C \rightarrow X$ with boundary in L is *adapted* to D iff

- (a) (Stable domain) The domain C is stable;
- (b) (Non-constant spheres) Each component of C that maps entirely to D is constant;
- (c) (Markings) Each interior marking z_i maps to D and each component of $u^{-1}(D)$ contains an interior marking.

Considering the the moduli space of *adapted* treed disks, we can prove a transversality result for *uncrowded* types. A combinatorial type is called *uncrowded* if each ghost component has at most one interior marking. This condition is necessary to prevent the expected dimension from running away to negative infinity.

First, the *combinatorial type* of a treed holomorphic disk $u : C \rightarrow X$ adapted to D is the combinatorial type Γ of the domain in addition to labelings $d : \text{Vert}(\Gamma) \rightarrow \pi^2(X) \sqcup \pi^2(X, L)$ recording the homotopy class of each disk/sphere, and $m : \text{Edge}_{\infty, s} \sqcup \text{Edge}_{<\infty, s} \rightarrow \mathbb{Z}_{\geq 0}$ recording the tangency of each spherical marking or node to the divisor.

The (Markings) axiom implies that for any spherical nodes that map to the divisor, u must be constant on one of the sphere/disk components. The order of tangency at this point is defined as the order on the non-constant component, or 0 if both sides are constant.

Let

$$\overline{\mathcal{U}}_{\Gamma}^{\text{thin}} \subset \overline{\mathcal{U}}_{\Gamma}$$

be an the an open neighborhood of the nodes and attaching points of the edges such that the compliment of the closure is open on each curve. Suppose that perturbation data $P_{\Gamma'}$ has been chosen for all boundary types $\mathcal{U}_{\Gamma'} \subset \overline{\mathcal{U}}_{\Gamma}$. Denote $\mathcal{P}_{\Gamma}^l(X, D)$ as the space of perturbation data $P_{\Gamma} = (F_{\Gamma}, J_{\Gamma})$ of class C^l equal to the given pair (F, J) on $\overline{\mathcal{U}}_{\Gamma}^{\text{thin}}$, and such that the restriction of P_{Γ} to $\mathcal{U}_{\Gamma'}$ is equal to $P_{\Gamma'}$ for each boundary type Γ' . Prescribing this equality gaurantees that the resulting collection satisfies the (Collapsing edges/Making edges finite or non-zero) axiom of the coherence condition. Let $\mathcal{P}_{\Gamma}(X, D)$ be the intersection of the spaces $\mathcal{P}_{\Gamma}^l(X, D)$ for all $l \geq 0$

For a partial ordering on combinatorial types of treed disks, we say

that $\Gamma' \leq \Gamma$ iff Γ is obtained from Γ' by (Collapsing edges/making edge lengths finite or non-zero).

Theorem 8. [4] (Transversality) *Suppose that Γ is an uncrowded type of stable treed marked disk of expected dimension $i(\Gamma, \underline{x}) \leq 1$. Suppose regular coherent perturbation data for types of stable treed marked disks Γ' with $\Gamma' \leq \Gamma$ are given. Then there exists a comeager subset $\mathcal{P}_\Gamma^{\text{reg}}(X, D) \subset \mathcal{P}_\Gamma(X, D)$ of regular perturbation data for type Γ compatible with the previously chosen perturbation data such that if $\mathcal{P}_\Gamma \subset \mathcal{P}_\Gamma^{\text{reg}}(X, D)$ then*

- (1) (Smoothness on each stratum) *The stratum $\mathcal{M}_\Gamma(L, D)$ is a smooth manifold of expected dimension.*
- (2) (Tubular neighborhoods) *If Γ is obtained from Γ' by collapsing an edge of $\text{Edge}_{<\infty, d}(\Gamma')$ or making an edge finite or non-zero or by gluing Γ' at a breaking, then the stratum $\mathcal{M}_\Gamma(L, D)$ has a tubular neighborhood in $\overline{\mathcal{M}}_\Gamma(L, D)$.*
- (3) (Orientations) *There exist orientations on $\mathcal{M}_\Gamma(L, D)$ compatible with the morphisms (Cutting an edge) and (Collapsing an edge/Making an edge finite or non-zero) in the following sense:*
 - (a) *If Γ is obtained from Γ' by (Cutting an edge) then the isomorphism $\mathcal{M}'_\Gamma(L, D) \rightarrow \mathcal{M}_\Gamma(L, D)$ is orientation preserving.*
 - (b) *If Γ is obtained from Γ' by (Collapsing an edge) or (Making an edge finite or non-zero) then the inclusion $\overline{\mathcal{M}}'_\Gamma(L, D) \rightarrow \overline{\mathcal{M}}_\Gamma(L, D)$ has orientation (from the decomposition*

$$T\mathcal{M}_\Gamma(L, D)|_{\mathcal{M}'_\Gamma(L, D)} \cong \mathbb{R} \oplus T\mathcal{M}'_\Gamma(L, D)$$

and the outward normal orientation on the first factor) given by a universal sign depending only on Γ, Γ' .

Proof. See [4] □

3.4. Compactness. We wish to have compactness of the 0 and 1 dimensional components of the moduli space $\overline{\mathcal{M}}_\Gamma(L, D)$ satisfying a certain energy bound. That is, we need to rule out bubbles mapping entirely to the divisor and unstable components.

Definition 13. [4] For $E > 0$, we say that an almost complex structure $J_D \in \mathcal{J}_\tau(X, D)$ is *E-stabilized* by a divisor D iff

- (a) (Non-constant spheres) D contains no non-constant J_D -holomorphic spheres of energy less than E .
- (b) (Sufficient intersections) each non-constant J_D -holomorphic sphere $u : S^2 \rightarrow X$ resp. J_D -holomorphic disk $u : (D, \partial D) \rightarrow (X, L)$

with energy less than E has at least three resp. one intersection points resp. point with the divisor D , that is, $u^{-1}(D)$ has order at least three resp. one.

Definition 14. [4] A divisor D with Poincaré dual $[D]^\wedge = k[\omega]$ for some $k \in \mathbb{N}$ has *sufficiently large degree* for an almost complex structure J_D iff

- $([D]^\wedge, \alpha) \geq 2(c_1(X), \alpha) + \dim(X) + 1$ for all $\alpha \in H_2(X, \mathbb{Z})$ representing non-constant J_D -holomorphic spheres.
- $([D]^\wedge, \beta) \geq 1$ for all $\beta \in H_2(X, L, \mathbb{Z})$ representing non-constant J_D -holomorphic disks.

Given $J \in \mathcal{J}_\tau(X, \omega)$ denote by $\mathcal{J}_\tau(X, D, J, \theta)$ as the space of tamed almost complex structures $J_D \in \mathcal{J}_\tau(X, \omega)$ such that $\|J_D - J\| < \theta$ in the sense of [7] and J_D preserves TD . We need the following lemma.

Lemma 2. [4] *For θ sufficiently small, suppose that D has sufficiently large degree for an almost complex structure θ -close to J . For each energy $E > 0$, there exists an open and dense subset $\mathcal{J}^*(X, D, J, \theta, E) \subset \mathcal{J}_\tau(X, D, J, \theta)$ such that if $J_D \in \mathcal{J}^*(X, D, J, \theta, E)$, then J_D is E -stabilized by D . Similarly, if $D = (D^t)$ is a family of divisors for J^t , then for each energy $E > 0$, there exists a dense and open subset $\mathcal{J}^*(X, D^t, J^t, \theta, E)$ in the space of time-dependent tamed almost complex structures $\mathcal{J}^*(X, D^t, J^t, \theta)$ such that if $J_{D^t} \in \mathcal{J}^*(X, D^t, J^t, \theta, E)$, then J_{D^t} is E -stabilized for all t .*

Let Γ be a type of stable treed disk, and let $\Gamma_1, \dots, \Gamma_l$ be the components formed by deleting boundary nodes of positive length, and $\overline{U}_{\Gamma_1}, \dots, \overline{U}_{\Gamma_l}$ the corresponding decomposition of the universal curve. Since $[D]^\wedge = k\omega$, any stable treed disk with domain of type Γ and transverse intersections with the divisor has energy at most

$$(5) \quad n(\Gamma_i, k) := \frac{n(\Gamma_i)}{k}$$

on the component \overline{U}_{Γ_i} , where $n(\Gamma_i)$ is the number of intersections of markings on \overline{U}_{Γ_i} with D .

Let $J_D \in \mathcal{J}_\tau(X, D, J, \theta)$ be an almost complex structure that is stabilized for all energies, (e.g., something in the intersection of $J_D \in \mathcal{J}^*(X, D, J, \theta, E)$ for all energies). For each energy E , there is a contractible open neighborhood $\mathcal{J}^{**}(X, D, J_D, \theta, E)$ of J_D in $J_D \in \mathcal{J}^*(X, D, J, \theta, E)$ that is E -stabilized.

Definition 15. A perturbation datum $P_\Gamma = (F_\Gamma, J_\Gamma)$ for a type of stable treed disk Γ is *stabilized* by D if J_Γ takes values in $\mathcal{J}^*(X, D, J, \theta, n(\Gamma_i, k))$ on \overline{U}_{Γ_i}

Theorem 9. (Compactness for fixed type) *For any collection $\underline{P} = (P_\Gamma)$ of coherent, regular, stabilized perturbation data and any uncrowded type Γ of expected dimension at most one, the moduli space $\overline{\mathcal{M}}_\Gamma(L, D)$ of adapted stable treed marked disks of type Γ is compact and the closure of $\mathcal{M}_\Gamma(L, D)$ contains only configurations with disk bubbling.*

Proof. See [4]. □

4. FLOER THEORY FOR FIBER BUNDLES.

We would like to use some of the previous scheme to help us achieve transversality for the moduli space of curves into certain *symplectic fibrations*. The class of symplectic fibrations that we will be working with is as follows:

Definition 16. A *symplectic Mori fibration* is a fiber bundle of symplectic manifolds $(F, \omega_F) \rightarrow (E, \omega) \xrightarrow{\pi} (B, \omega_B)$, where (F, ω_F) is monotone, (B, ω_B) is rational, and $\omega = a + K\pi^*\omega_B$ for large K with $\iota^*a = \omega_F$.

Definition 17. A *fibred Lagrangian* is a Lagrangian in a symplectic Mori fibration $L \subset E$ such that there are Lagrangians $L_F \subset F$ and $L_B \subset B$ and π induces a fiber bundle $L_F \rightarrow L \rightarrow L_B$

In general, the Floer cohomology of L_B may not be defined due to bubbling. However, the usual transversality and compactness should still hold for L if we combine these technical results for L_F and L_B . On the other hand, our primary interest is in $L \subset E$ which is neither monotone nor part of a rational symplectic manifold, so we take care in this section to make sure that the usual results hold. In a nutshell, we pull back the divisor from the base to stabilize Floer trajectories which intersect fibers transversely, and use the usual monotone results for pseudo holomorphic curves which lie completely in a fiber.

4.1. Divisors. To use the perturbation scheme from Chapter 3, we pick a divisor in B and take its inverse image under π to get a divisor in E . Let $\theta > 0$, and suppose J_H is a compatible almost complex structure on the bundle $H \rightarrow E$. An almost complex structure on this bundle is called *basic* if it is $\pi^*(K)$ for some almost complex structure on B . We will achieve transversality by using domain dependent a.c. structures of the form $J_B + J_{ut}$, where the J_B is basic on H and J_{ut} is upper triangular with respect to the connection. We begin by choosing a divisor $D_B \subset B$ which is weakly stabilizing for L_B with respect to a taming a.c. structure J_{D_B} which makes D_B into an a.c. submanifold. The existence of such structure is guaranteed by [7, 5] and others, and is summarized (for our purposes) in Theorem 7. The pair $(\pi^{-1}(D_B), J_{ut})$

form an a.c. symplectic submanifold for sufficiently K in the weak coupling form and J_{ut} upper triangular w.r.t. the connection which agrees with $\pi^*J_{D_B}$ on H .

Definition 18. We will say that a Floer trajectory u is π -adapted to D if $\pi \circ u$ is adapted to $\pi(D)$ in the sense of definition 12:

- (a) The domain C is stable after collapsing any component on which u is non-constant but on which $\pi \circ u$ is constant;
- (b) Each component of C that maps entirely to D_B is constant;
- (c) Each interior marking z_i maps to D and each component of $u^{-1}(D)$ contains an interior marking.

Definition 19. A divisor D is stabilizing for L if it is the inverse image of a stabilizing divisor D_B for L_B in sense of definition 11:

There exists an almost-complex structure $J_{D_B} \in (\mathcal{J}, \omega_B)$ adapted to D_B such that any J_{D_B} holomorphic disk $u : (C, \partial C) \rightarrow (B, L_B)$ with $\omega_B([u]) > 0$ intersects D_B in at least one point.

We label an associated a.c. structure (which makes D into an a.c. submanifold) J_D .

4.2. Perturbation Data. One of the components of the input data requires the choice of a base Morse-Smale function on L . It will be important later on that we choose the function so that it decends to a datum on B . We can construct a Morse function on L by the following recipe: take Morse functions b resp. g on L_B resp. L_F . Take trivializations $\{(U_i, \Psi_i)\}$ with the U_i small neighborhoods of the critical points $\{x_i\}$ for b . Let ϕ_i be bump functions equal to 1 in a neighborhood of each x_i and 0 outside U_i . The function $f = \pi^*b + \sum_i \pi^*\phi_i\Psi^*g$ is then a Morse function for L with the property that its restriction fibers near the critical points is also Morse. This function can then be perturbed in a finite number of small neighborhoods outside of the critical points to make it Morse-Smale.

Definition 20. An M -type perturbation datum for $(F \rightarrow E \rightarrow B, \omega)$, denoted $\mathcal{P} = \{P_\Gamma\}_\Gamma$, is a family of $\mathcal{U}_\Gamma \rightarrow \mathcal{J}_{ut}^l \oplus C^l(L)$ where the first factor is upper triangular $J_{ut} = \begin{bmatrix} J_{TF} & J_H \\ 0 & J_B \end{bmatrix}$ with respect to the connection and J_B equal to J_D in a neighborhood of the interior markings, spherical nodes, and on the boundary component of each disk. The

second factor is required to be equal to f in a neighborhood of ∞ and boundary disk markings.

For an even dimensional real vector space V , the space $\mathcal{J}_{ut}^l(V)$ can be viewed as a (trivial) vector bundle $\mathcal{K}^l \rightarrow \mathcal{J}_{ut}^l \rightarrow \mathcal{J}_F^l \oplus \mathcal{J}_B^l$, where the base are the bundles of a.c. structures on F and B respectively. Consider $\mathcal{J}_{ut}^l(V)$ for a fixed $2m + 2n$ dimensional vector space $V = X \oplus Y$. Given a.c. structures (J, K) , the set of $m \times n$ matrices L which make $\begin{bmatrix} J & L \\ 0 & K \end{bmatrix}$ into an a.c structure satisfy the linear relation $JL + LK = 0$.

For J_0 resp. K_0 in normal form $\begin{bmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{bmatrix}$, this is the set of $2n \times 2m$ matrices $\begin{bmatrix} A & B \\ B & -A \end{bmatrix}$ as one can check. The set of a.c. structures on X resp. Y are given by the homogeneous space $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$ resp. $GL(2m, \mathbb{R})/GL(m, \mathbb{C})$. For $J = CJ_0C^{-1}$, $K = DK_0D^{-1}$, the fiber at (J, K) is given by the above form conjugated by C, D^{-1} . For a contractible open neighborhood U of (J, K) , choose a section of the bundle $GL(2n, \mathbb{R}) \times GL(2m, \mathbb{R}) \rightarrow \mathcal{J}_X \times \mathcal{J}_Y$. This gives a smooth choice of coset representatives $([A], [B]) \mapsto (s_1(A), s_2(B)) \in A \cdot G_{K_0} \times B \cdot G_{J_0}$. Thus define a local trivialization of $\mathcal{J}_{ut}^l(V)$ on U by $([A], [B], L) \mapsto ([A], [B], s_1^{-1}(A) \cdot L \cdot s_2(B))$. Transition maps for a choice of section $([A], [B]) \mapsto (t_1(A), t_2(B))$ over an intersecting V are given by $L \mapsto t_1^{-1}(A)s_1(A) \cdot L \cdot s_2^{-1}(B)t_2(B)$.

Furthermore, given a symplectic form ω on V with $V = X \oplus Y$, $Y = X^\omega$, $\mathcal{J}_{ut,\tau}^l(V, \omega)$ is the open set of upper triangular a.c. structures which tame ω . Notice that this is merely the above vector bundle restricted to the base $\mathcal{J}_{X,\tau}^l(\omega) \oplus \mathcal{J}_{Y,\tau}^l(\omega)$, which is a contractible space, and so gives a trivial vector bundle.

In general the space $\mathcal{J}_{ut,\tau}^l(E, \omega)$ is a banach manifold which can be realized as a banach vector bundle $\mathcal{J}_A \rightarrow \mathcal{J}_{ut,\tau}^l(E, \omega) \rightarrow \mathcal{J}_{TF,\tau}^l \oplus \mathcal{J}_{B,\tau}^l$, where the fiber at a point (J_F, J_B) is the space of sections over E such that $J_F J_H + J_H J_B = 0$. Thus the tangent space to a point (J_F, J_B, J_A) is given by the set of matrices $\begin{bmatrix} K_F & K_A \\ 0 & K_B \end{bmatrix}$ where the K_\bullet anti-commute with their respective a.c. structure and K_H satisfies $J_F K_A + K_A J_B = 0$.

It should be noted that the space of such upper triangular structures that are adapted the the divisor is still a banach vector bundle, as this

only imposes a constraint on the base part of the structure.

4.2.1. *Coherence and π -stability.* The type of requirements that we need for stability are slightly more delicate than those in the strictly rational case.

As in section 3, the *combinatorial type* Γ of a treed disk will contain the following information:

- (1) the set of vertices, edges, edges lengths, and node/marking type where edges meet vertices
- (2) the homotopy class that each vertex is required to represent as a domain for a map u
- (3) the tangency of each interior marking to the divisor $\pi^{-1}(D_B)$ along the connection H
- (4) a binary marking which dictates how each vertex (as a disk/sphere domain) behaves w.r.t. π (see below)

Definition 21. A *binary marking* or *coloring* of a combinatorial type Γ is a subset of the vertices and edges, denoted mv resp. me , for which any map $u : \mathcal{U}_\Gamma \rightarrow E$ is required to map the domain for mv to a constant under π resp. the domain for me to a constant under π . The set of unmarked vertices and edge will be denoted uv resp. ue .

Definition 22. A combinatorial type is called *π -stable* if each unmarked vertex uv is stable.

Definition 23. A *coherent* collection of M-type perturbation datum $\{(J_{\Gamma,ut}^l(E, \omega), f_\Gamma)\}_\Gamma$ for π -stable types is one with the following properties:

- (1) $J_{\Gamma,TF}$ is constant on each surface component of the universal treed disk \mathcal{U}_Γ
- (2) On marked vertices, all perturbation data is constant on the corresponding surface component
- (3) If Γ' is obtained from Γ by forgetting a marked vertex and stabilizing the domain, then the perturbation data P_Γ agrees with the pullback of $P_{\Gamma'}$ under the natural map of universal disks.
- (4) The collection $\{(J_{\Gamma,B}^l, f_\Gamma)\}_\Gamma$ obeys the axioms for a coherent perturbation system from the rational case (8).

4.3. **Transversality.** In the fibered situation, we say that a Floer trajectory $u : C \rightarrow E$ is *adapted* to D if $\pi \circ u$ is adapted to D_B in sense of definition 12 after removing any marked vertices. Denote by

$\mathcal{M}_\Gamma(E, D, P)$ the moduli space of type Γ Floer trajectories which are adapted to D with respect to some perturbation data P , and for a tuple (x_0, \dots, x_n) , by $\mathcal{M}_\Gamma(E, D, P, \bar{x})$ the ones which limit to x_0 along the root and (x_1, \dots, x_n) along the leaves, arranged in counterclockwise order.

The expected dimension of the stratum $\mathcal{M}_\Gamma(E, D, P, \bar{x})$ is

$$\begin{aligned} \iota(\Gamma, \bar{x}) := & \dim W_f^+(x_0) - \sum_{i=1}^n \dim W_f^+(x_i) + \sum_{i=1}^n I(u_i) + n - 2 - |\text{Edge}_{<\infty}^0(\Gamma)| \\ & - |\text{Edge}_\infty(\Gamma) - (n+1)|/2 - 2|\text{Edge}_{<\infty, s}(\Gamma)| - |\text{Edge}_{\infty, s}(\Gamma)| \\ & - \sum_{e \in \text{Edge}_{\infty, s}} m(e) - \sum_{e \in \text{Edge}_{<\infty, s}} m(e). \end{aligned}$$

Let $\overline{\mathcal{S}}_\Gamma \subset \overline{\mathcal{U}}_\Gamma$ be the two-dimensional part of the universal treed disk, and $\overline{\mathcal{T}}_\Gamma \subset \overline{\mathcal{U}}_\Gamma$ be the tree part of the universal treed disk. Fix a compact set

$$\overline{\mathcal{S}}_\Gamma^o \subset \overline{\mathcal{S}}_\Gamma$$

not containing the boundary, nodes, or interior markings, but having non-trivial intersection with every fiber of the universal disk \mathcal{U}_Γ . Also fix a compact set

$$\overline{\mathcal{T}}_\Gamma^o \subset \overline{\mathcal{T}}_\Gamma$$

having non-trivial intersection with each universal fiber. Thus, the compliments

$$\begin{aligned} \overline{\mathcal{S}}_\Gamma - \overline{\mathcal{S}}_\Gamma^o \\ \overline{\mathcal{T}}_\Gamma - \overline{\mathcal{T}}_\Gamma^o \end{aligned}$$

are neighborhoods of the boundary, interior markings, and nodes resp. neighborhoods of ∞ in each fiber of the universal disk. We require that the perturbation data vanish in these neighborhoods. In addition, we only consider types Γ which are uncrowded.

We say that a type $\Gamma' \leq \Gamma$ iff Γ is obtained from Γ' by (collapsing edges/making an edge length finite or non-zero).

Theorem 10 (Transversality). *Suppose Γ is an uncrowded combinatorial of expected dimension $\iota(\Gamma, \bar{x}) \leq 1$. Suppose that a coherent system of perturbation data has been chosen for all types $\Gamma' \leq \Gamma$. Then there is a comeager subset of M-type perturbation datum $\mathcal{P}_\Gamma^{\text{reg}}(E, D) \subset \mathcal{P}_\Gamma(E, D)$, which is compatible with the previously chosen data, such that the following hold:*

- (1) *The moduli space $\mathcal{M}_\Gamma(D, P)$ for $P \in \mathcal{P}_\Gamma^{\text{reg}}$ is a smooth manifold of expected dimension.*

- (2) *The (orientations) and (tubular neighborhoods) statements from theorem 8 hold.*

Proof. The proof follows some of the ideas in [4] in addition to making special choices of perturbation data for the fiber and upper triangular part. If C is a nodal disk of type Γ , for $p \geq 2$ and $k > 2/p$ let $\text{Map}^0(C, E, L)_{k,p}$ denote the space of (continuous) maps from C to E with boundary and edge components in L which are of the class $W^{k,p}$ on each disk, sphere, and edge. We have the following standard result:

Lemma 3. $\text{Map}^0(C, E, L)_{k,p}$ is a Banach manifold with local charts centered at u given by the product space of vector fields which agree at disk nodes and interior markings:

$$\bigoplus_{(v,e) \in \Gamma} W^{k,p}(C, u_v^* TE, u_{v,\partial C}^* TL) \oplus_{\text{Edge}_d} W^{k,p}(C, u_e^* TL)$$

where the map into Map^0 is given by geodesic exponentiation with respect to some metric on E which makes L and D totally geodesic.

Let $\text{Map}_\Gamma^0(C, E, L)_{k,p} \subset \text{Map}^0(C, E, L)_{k,p}$ denote the submanifold of maps whose spheres and disks map to the labeled homology classes which have the prescribed tangencies to the divisor, and whose marked vertices are constant with respect to π .

In general, the space $\text{Map}_\Gamma^0(C, E, L)_{k,p}$ is a C^q Banach submanifold where $q < k - n/p - \max_e m(e)$. Following Dragnev [9], the universal space is constructed as follows. Given a trivialization of the universal disk $C \in U_\Gamma^i \rightarrow \mathfrak{M}_\Gamma^i$, we get a map $m \mapsto j(m) \in \mathcal{J}(S)$ obtained by identifying nearby curves with C . Consider the product space

$$(6) \quad \mathcal{B}_{k,p,\Gamma,l}^i := \mathfrak{M}_\Gamma^i \oplus \text{Map}_\Gamma^0(C, E, L)_{k,p} \oplus \mathcal{P}_\Gamma^l(E, D).$$

Over this Banach manifold we get a vector bundle $\mathcal{E}_{k,p,\Gamma,l}^i$ given by

$$(7) \quad (\mathcal{E}_{k,p,\Gamma,l}^i)_{m,u,J,F} \subset \bigoplus_{v,e \in \Gamma} \text{Map}_{j,J,\Gamma}^{0,1}(C, u_v^*(TF \oplus H))_{k-1,p}$$

$$(8) \quad \oplus \text{Map}^1(C, u_e^* TL)_{k-1,p}$$

the space of $(0, 1)$ -forms and 1-forms over C with values in the indicated vector bundle which vanish to order $m(e) - 1$ at the node or marking corresponding to e (hence the \subset and not equality). Local trivializations of this bundle are given by parallel transport along geodesics in E via the associated hermitian connection in the fibers. For the transition maps to be C^q , we need l large so that $q < l - k$.

There is a C^q section $\bar{\partial} : \mathcal{B}_{k,p,\Gamma,l}^i \rightarrow \mathcal{E}_{k,p,\Gamma,l}^i$ via

$$(9) \quad (m, u, J, F) \mapsto (\bar{\partial}_{j(m),J} u_S, (\frac{d}{ds} - \text{grad}_f) u_T)$$

with

$$(10) \quad \bar{\partial}_{j(m),J} u_S := du_S + J \circ du_S \circ j(m)$$

The a.c. structure J depends on $(m, p) \in \mathfrak{M}_\Gamma^i \oplus C$. The *local universal moduli space* is defined to be

$$(11) \quad \mathcal{M}_\Gamma^{univ,i}(E, L, D) := \bar{\partial}^{-1} \mathcal{B}_{k,p,\Gamma,l}^i$$

where $\mathcal{B}_{k,p,\Gamma,l}^i$ is identified with the zero section.

Surjectivity on the edges is a matter of a standard argument.

With respect to the variable in $\text{Map}_\Gamma^0(C, E, L)_{k,p}$, the linearization of the Cauchy Riemann operator $\bar{\partial}_{j(m),J}$ is

$$(12) \quad D_{u,J,j}(\xi) = \nabla \xi + J \circ \nabla \xi \circ j - J(u)(\nabla_\xi J) \partial_{j(m),J} u_s$$

We also have zeroth order terms coming from the domain dependent data: The differential of $\bar{\partial}_{j(m),J}$ at a J_Γ holomorphic map w.r.t. the variable in $\mathcal{P}_\Gamma(E, D)$ is given by

$$(13) \quad T_{J_\Gamma} \mathcal{P}_\Gamma \rightarrow \text{Map}^{(0,1)}(C, u^* T E)_{k-1,p}, \quad K \mapsto K \circ du_S \circ j$$

The surjectivity argument for this divided into multiple cases: given a component u_v of a Floer trajectory, the component can either be constant in the horizontal direction, the vertical direction, both, or neither. Notably, we have the splittings of the domain of the linearized Cauchy-Riemann operator:

$$\begin{aligned} D_u(K) : & W^{k,p}(S, u^* T F, u_{\partial S}^* T F \cap T L) \oplus W^{k,p}(S, u^* H, u_{\partial S}^* H \cap T L) \\ & \rightarrow \text{Map}_{j,J,\Gamma}^{0,1}(S, u^* T F \oplus H)_{k-1,p} \end{aligned}$$

While the range does not split in such a manner (unless $J_H \equiv 0$), we have the nice feature of additional freedom in the choice of perturbation data. Now, supposing that u is J -holomorphic, D_u restricts to a map

$$(14) \quad D_u : W^{k,p}(S, u^* T F, u_{\partial S}^* T F \cap T L) \rightarrow \text{Map}_{j,J,\Gamma}^{0,1}(S, u^* T F)_{k-1,p}$$

By construction, any J -holomorphic disk/sphere u gives rise to a J_B holomorphic disk/sphere $\pi \circ u$. We use this fact in each of the following 3 cases:

Case 1: u is only constant in the horizontal direction.

In this case, the domain corresponds to a marked vertex of Γ . In the vertical direction, we have that $du_F \circ j = J_{TF} du_F$ since the horizontal differential vanishes. Thus, u is a J_{TF} -holomorphic curve in the monotone manifold F_p (with boundary conditions in $F_p \cap L$ in the disk case). First, assume that u is simple. In this case, we use the standard argument from [25] to get surjectivity for the restriction of the linearized operator in (14).

Now suppose u is a disk component but not simple. Then by decomposition results due to [21], we have that u represents a sum of elements of $H_2(E, L)$. If $\dim F \geq 3$, we must have that $I(u) = mI(\tilde{u})$, where $\tilde{u} \circ p = u$ for simple J -holomorphic \tilde{u} and holomorphic covering map p . Replacing u with \tilde{u} in the configuration Γ gives a simple configuration $\tilde{\Gamma}$, which can be made regular by the above paragraph. Since $I(\tilde{u}) \geq 2$ and $\iota(\tilde{\Gamma}, \bar{x}) \geq 0$, we must have had that $\iota(\Gamma, \bar{x}) \geq 2$, which is a contradiction. The case when $\dim F = 2$ is similar, see [3].

If u is a non-constant and nowhere injective sphere component attached to a configuration $\tilde{\Gamma}$, then we must have that $u = \tilde{u} \circ p$ for a degree $d > 1$ branched covering map p . From this, we get that $2c_1(A_u) = 2dc_1(A_{\tilde{u}}) > 0$ since u is non-constant and F is monotone. The configuration Γ with u replaced by \tilde{u} is regular by the above paragraph, and so it has expected dimension. This gives us that Γ with the map u must be of index ≥ 2 , which goes against the assumption.

Pick a J invariant complement to u^*TF (for instance, $u^*TF \oplus X$, where $X = \{v + (J_H \circ u)v \mid v \in u^*H\}$). Then there is a J equivariant short exact sequence which lifts the identity on S :

$$(15) \quad \begin{aligned} 0 \rightarrow \text{Map}_{j,J,\Gamma}^{0,1}(S_{mv}, u^*TF) &\rightarrow \text{Map}_{j,J,\Gamma}^{0,1}(S_{mv}, u^*(TF \oplus H)) \\ &\rightarrow \text{Map}_{j,J,\Gamma}^{0,1}(S_{mv}, X) \rightarrow 0 \end{aligned}$$

X projects isomorphically onto u^*TB . Thus, $X \cong (\pi \circ u)^*TB$ J -equivariantly, which is the trivial bundle. By surjectivity for constant curves, we have that

$$(16) \quad D_u : W^{k,p}(S_{mv}, u^*X, X \cap u_{\partial S_{mv}}^* TL) \rightarrow \text{Map}_{j,J,\Gamma}^{0,1}(S_{mv}, X)_{k-1,p}$$

is surjective. Since $\text{Map}_{j,J,\Gamma}^{0,1}(S, u^*TF \oplus H) \cong \text{Map}_{j,J,\Gamma}^{0,1}(S, u^*TF) \oplus \text{Map}_{j,J,\Gamma}^{0,1}(S, X)$ (and similarly for $W^{k,p}$ sections), we have transversality in this case.

Case 2: u is only constant in the vertical direction.

Similar to the above case, we have an equivariant splitting $\text{Map}_{j,J,\Gamma}^{0,1}(S_{uv}, u^*(TF \oplus H)) \cong \text{Map}_{j,J,\Gamma}^{0,1}(S_{uv}, u^*TF) \oplus \text{Map}_{j,J,\Gamma}^{0,1}(S_{uv}, X)$.

To get surjectivity onto the first summand we leverage the upper triangular part of the a.c. structure. First consider the case when u has no tangencies to the divisor. Following the type of argument in [25], we prove that the image of the linearized map is dense in $\text{Map}^{0,1}(S_{uv}, u^*TF)_{k-1,p}$. Suppose that the image is not dense. Since this is a Fredholm operator, the image is closed. By the Hahn-Banach theorem, there is a non-zero element $\eta \in \text{Map}^{0,1}(S_{uv}, u^*TF)_{k-1,q}$ such that

$$(17) \quad \int_C \langle D_u^{TF} \xi + K \circ du_H \circ j, \eta \rangle = 0$$

for every $\xi \in \text{Map}^0(S_{uv}, u^*TF)_{k-1,p}$ and K with $J_F K + K J_B = 0$. Thus, we have the following identities:

$$(18) \quad \int_C \langle D_u^{TF} \xi, \eta \rangle = 0$$

$$(19) \quad \int_C \langle K \circ du_H \circ j, \eta \rangle = 0$$

It follows [25] that η is a solution the Cauchy-Riemann type equation

$$D_u^{TF*} \eta = 0$$

where D_u^{TF*} is the formal adjoint. Thus, η is of class $(k-1, q)$, and it follows that $\eta \neq 0$ on a dense subset of S_{uv} .

Lemma 4. *Let $0 \neq \eta \in Y$ and $0 \neq \xi \in X$ with corresponding a.c. structures J_Y resp. J_X . Then there is a K with $J_Y K J_X = K$ such $K\xi = \eta$*

Proof. This requires us to find a complex anti-linear K such that $K\xi = \eta$, which is straightforward. See [25]. \square

Pick a point p where $du_H \neq 0 \neq \eta$ which is contained in the complement of $\overline{\mathcal{U}}_\Gamma^{\text{thin}}$. Then there is a $K_0 \in T_{J_{u(p)}} \mathcal{J}$ such that $\langle K_0 \circ du_{H,p} \circ j, \eta(p) \rangle > 0$. From the perturbation data $J_\Gamma : C \rightarrow \mathcal{J}_{ut}(\omega, D)$, we construct a section $K_\Gamma : C \rightarrow T_{J_\Gamma} \mathcal{J}_{ut}$ such that $K_\Gamma(p, u(p)) = K_0$ and K_Γ is supported in a sufficiently small neighborhood $U \times V$ with u injective on U and $\langle K_\Gamma(x, u(x)) \circ$

$du_{H,x} \circ j, \eta(x)\rangle > 0$ whenever $K_\Gamma(x, u(x)) \neq 0$. We must then have that

$$\int_C \langle K \circ du_H \circ j, \eta \rangle > 0$$

which is a contradiction. Therefore, the linearized operator must be surjective onto the TF part of the summand in this case.

When there are tangencies to the divisor, the above method in combination with Lemma 6.6 from [7] gives surjectivity.

For the X part of summand, we use the fact that $X \cong u^*H$ via a (J, J_B) equivariant map. The later is isomorphic to $(\pi \circ u)^*TB_{J_B}$ equivariantly. Therefore we have $\text{Map}_{j,J}^{(0,1)}(C, X)_{(k-1,p)} \cong \text{Map}_{j,J_B}^{(0,1)}(C, (\pi \circ u)^*TB)_{(k-1,p)}$ (and easier: $\text{Map}^0(C, X)_{(k,p)} \cong \text{Map}^0(C, \pi \circ u^*TB)_{(k,p)}$).

Thus, surjectivity of the map $D_u : \text{Map}^0(C, X)_{k,p} \rightarrow \text{Map}_{j,J_B}^{(0,1)}(C, X)$ follows from the techniques in [4]. Specifically, the expected dimension of the stratum containing the map $\pi \circ u$ is not greater than that containing u . Thus, the techniques to prove surjectivity at $\pi \circ u$ from theorem 2.18 [4] can be used. This concludes the proof of transversality in case 2.

Case 3: $du_H, du_F \neq 0$

Surjectivity onto the X part of the summand is the same as Case 2.

The map $D_u : \text{Map}^0(C, u^*TF) \rightarrow \text{Map}_{j,J}^{(0,1)}(C, u^*TF)$ is surjective for the same reasons that it is in case 2: if u is multiply covered, then we can use the domain dependant upper triangular part of the a.c. structure to achieve transversality.

By the implicit function theorem, this universal moduli space is a C^q Banach manifold.

The general theory of real Cauchy-Riemann operators [25] tells us that the linearization $D_u + K \circ du \circ j$ is Fredholm, so has finite dimensional kernel. We now consider the restriction of the projection $\Pi : \mathcal{B}_{k,p,\Gamma,l}^i \rightarrow \mathcal{P}_\Gamma^l(E, D)$ to the universal moduli space. The kernel and cokernel of this projection are isomorphic the kernel and cokernel of the operator D_u , respectively. Thus, Π is a Fredholm operator with the same index as D_u . Let $\mathcal{M}_d^{univ,i}$ be the component of the universal space on which Π has Fredholm index d . By the Sard-Smale theorem, for q large enough, the set of regular values of Π , $\mathcal{P}_\Gamma^{l,reg}(E, D)_{d,i}$, is comeager. Let

$$\mathcal{P}_\Gamma^{l,reg}(E, D)_d = \bigcap_i \mathcal{P}_\Gamma^{l,reg}(E, D)_{d,i}$$

Then this is also a comeager set. An argument due to Taubes (see [25]) shows that the set of smooth regular perturbation datum

$$\mathcal{P}_\Gamma^{reg}(E, D)_d = \bigcap_l \mathcal{P}_\Gamma^{l, reg}(E, D)_d$$

is also comeager. For $P_\Gamma = (J_\Gamma, G_\Gamma)$ in the set of smooth regular data, notate $\mathcal{M}_\Gamma^i(E, L, D, P_\Gamma)$ as the space of P_Γ trajectories in the trivialization i , a C^q manifold of dimension d . By elliptic regularity, every element of $\mathcal{M}_\Gamma^i(E, L, D, P_\Gamma)$ is smooth. Using the transition maps for the universal curve of Γ , we get maps $g_{ij} : \mathcal{M}_\Gamma^i \cap \mathcal{M}_\Gamma^j \rightarrow \mathcal{M}_\Gamma^i \cap \mathcal{M}_\Gamma^j$ which serve as transition maps for the space

$$\mathcal{M}_\Gamma(E, L, D, P_\Gamma) = \bigcup_i \mathcal{M}_\Gamma^i(E, L, D, P_\Gamma)$$

Since each piece $\mathcal{M}_\Gamma^i(P_\Gamma)$ and the moduli space of treed disks is Hausdorff and second countable and the moduli space of treed disks is, it follows that $\mathcal{M}_\Gamma(P_\Gamma)$ is Hausdorff and second countable.

The gluing argument that produces the tubular neighborhood of $\mathcal{M}_\Gamma'(E, L, D, P)$ in $\mathcal{M}_\Gamma(E, L, D, P)$ is the same as in [5, 4]. The matter of assigning compatible orientations is also expected to be similar. \square

4.4. Compactness. The main goal of this section is to establish the compactness of the moduli space $\overline{\mathcal{M}}_\Gamma(E, L, D, P)$ for a coherent system of regular perturbation datum. We use the existence of a divisor D_B and an appropriate choice of perturbation data to rule out sphere bubbling in the base, and then complete the result with well known facts about compactness in monotone symplectic manifolds.

Definition 24. For a divisor $D = \pi^{-1}(D_B)$, we say that an adapted (upper triangular) a.c. structure J with basic lower block diagonal J_{D_B} is *e-stabilized* by D if J_{D_B} is *e-stabilized* by D_B as in definition 13:

- (a) (Non-constant spheres) D_B contains no non-constant J_{D_B} -holomorphic spheres of energy less than e .
- (b) (Sufficient intersections) each non-constant J_{D_B} -holomorphic sphere $u : S^2 \rightarrow B$ resp. J_{D_B} -holomorphic disk $u : (D, \partial D) \rightarrow (B, L_B)$ with energy less than e has at least three resp. one intersection points resp. point with the divisor D_B , that is, $u^{-1}(D_B)$ has order at least three resp. one.

Definition 25. We say that D is of large enough degree for an adapted J if D_B is for J_{D_B} as in definition 14:

- (1) $([D_B]^\wedge, \alpha) \geq 2(c_1(B), \alpha) + \dim(B) + 1$ for all $\alpha \in H_2(B, \mathbb{Z})$ representing non-constant J_{D_B} -holomorphic spheres.
- (2) $([D_B]^\wedge, \beta) \geq 1$ for all $\beta \in H_2(B, L_B, \mathbb{Z})$ representing non-constant J_{D_B} -holomorphic disks.

A similar result holds as in Lemma 2 for a dense open set which is e stabilizing. Indeed, suppose we have a basic a.c. structure J_{D_B} for which D_B is of sufficiently large degree and is θ -close to J_B . There is an open, dense set $\mathcal{J}_\tau^*(B, D_B, J_B, \theta, e) \subset \mathcal{J}_\tau(B, D_B, J_B, \theta)$ given by Lemma 2. To get a collection of upper triangular e -stabilizing a.c. structures on E , we take the inverse image of this set under the projection $\pi : \mathcal{J}_{ut, \tau} \rightarrow \mathcal{J}_{B, \tau}$. We shall denote the (dense, open) set obtained in this manner $\mathcal{J}_\tau^*(E, D, J_B, \theta, e)$.

For a π -stable combinatorial type Γ , let $\Gamma_1, \dots, \Gamma_l$ be the decomposition obtained by deleting boundary nodes of positive length, and further requiring that each component only contains marked or unmarked vertices. Let $\overline{U}_{\Gamma_1}, \dots, \overline{U}_{\Gamma_l}$ the corresponding decomposition of the universal curve. Since $[D_B]^\wedge = k\omega_B$, any stable treed holomorphic disk projected to B with domain of unmarked type Γ_i and transverse intersections with the divisor has energy at most

$$(20) \quad n(\Gamma_i, k) := \frac{n(\Gamma_i)}{C(k)}$$

on the component \overline{U}_{Γ_i} , where $n(\Gamma_i)$ is the number of markings on \overline{U}_{Γ_i} and $C(k)$ is an increasing linear function of k arising in the construction of D_B in [7].

Definition 26. A perturbation datum $P_\Gamma = (F_\Gamma, J_\Gamma)$ for a type of stable treed disk Γ is *stabilized* by D if J_Γ takes values in $\pi^{-1}\mathcal{J}_\tau^*(B, D_B, J_B, \theta, n(\Gamma_i, k))$ on \overline{U}_{Γ_i}

We now prove the main theorem of this section:

Theorem 11. *For any collection $\underline{P} = (P_\Gamma)$ of coherent, regular, stabilized perturbation data and any uncrowded type Γ of expected dimension at most one, the moduli space $\overline{\mathcal{M}}_\Gamma(L, D)$ of π -adapted stable treed marked disks of type Γ is compact and the closure of $\mathcal{M}_\Gamma(L, D)$ only contains configurations with disk bubbling.*

Proof. It is enough to check sequential compactness. Let Γ be a connected, uncrowded combinatorial type (which is stable on unmarked

vertices), and let $u_\nu : C_\nu \rightarrow E$ be a sequence of J_Γ -holomorphic maps. We decompose Γ into partial sub-types Γ_i by cutting finite length edges which connect marked vertices to unmarked ones. The proof will be in cases.

Case 1: Γ_i is an unmarked partial subtype.

Since we are on an unmarked subtype, the π -adapted Floer trajectories are actually *adapted* to D in the sense of [4]. The sequence $u^\nu : C_i^\nu \rightarrow E$ has a Gromov-Floer limit $u : C_i' \rightarrow E$ for a possibly unstable curve class $[\hat{C}]$ with stabilization $[C]$. Since $\pi(u^\nu) \mapsto \pi(u)$, the fact that u is π -adapted follows from [4]. We include the argument here for completeness' sake.

Since $J_\Gamma = J_D \in \mathcal{J}_\tau^*(B, D_B, J_B, \theta, n(\Gamma_i, k))$ over D , D_B contains no $\pi_* J_D$ -holomorphic spheres from $\pi(u)$. Thus, the (non-constant spheres) property.

Any unstable disk component u_j in the limit would be J_D -holomorphic. Unless it is constant, $\pi \circ u_i$ would be J_{D_B} -holomorphic and have at least one intersection with D_B by the stabilizing property of D_B . Thus, unstable disk components can only occur in the vertical direction.

Similarly, suppose we have a non-constant unstable sphere component u_j . Then $\pi \circ u_j$ has energy at most $n(\Gamma_i, k)$ since it is the limit of types with energy bounded by this. Since $J_\Gamma = J_D$ on $\pi \circ u_j$, there must be at least three intersection points with D_B on this component, unless $\pi \circ u_j$ is constant. Thus, unstable sphere components only occur in the vertical direction.

Therefore, for an unmarked sub-type, the only additional thing that we can pick up is a marked component (in the vertical direction). We argue that this cannot occur:

If we have a vertical sphere bubble v from an unmarked disk, then it must have positive energy and hence positive chern number. The limiting configuration is regular by appropriate choice of coherent perturbation data, and by the coherence condition we also have regularity for the configuration without the sphere bubble, so both are of expected dimension. On the other hand, the expect dimension of each of these types differs by 2 (by the presence of a spherical node), which contradicts the index assumption.

The case against a vertical disk bubble is the same as the argument against a disk bubble in the strictly monotone case: see below.

Case 2: Γ_i is a marked subtype.

By construction, the Morse-Smale function f restricted to any critical fiber is Morse-Smale. Thus, for a connected unmarked subtype mapping to a critical fiber, we are considering Morse-Floer trajectories on a monotone Lagrangian $L_{F_b} \subset F_b$. Away from the critical fibers, the flow lines intersect the fibers transversely, so the only marked configurations contained in non-critical fibers are nodal-disks with zero length edges.

The index formula that gives us that the dimension of the open strata for an admissible set of critical points (x_0, \dots, x_n) , after modding out by isomorphism, is:

$$\begin{aligned} \iota(\Gamma, \bar{x}) := & \dim W_F^+(x_0) - \sum_{i=1}^n \dim W_F^+(x_i) + \sum_{i=1}^n I(u_i) + n - 2 \\ & - |\text{Edge}_{<\infty}^0(\Gamma)| - |\text{Edge}_{\infty}(\Gamma) - (n+1)|/2 - 2|\text{Edge}_{<\infty,s}(\Gamma)| \end{aligned}$$

where $I(u_i)$ is either the Maslov index of u_i or $2c(A_i)$ with A_i as the spherical homology class of u_i . By the monotone property of (F_p, L_p) , we can replace this term with $\lambda\omega_F(u_i)$ if we so choose. For a fixed energy, Gromov compactness gives us a subsequence which Gromov-Floer converges to a limiting treed holomorphic treed disk u of the same energy. First assume that the limiting configuration Γ contains a non-constant sphere bubble. The index of the linearized operator is preserved under limits (see [26]), this configuration is of expected dimension ≤ 1 , and thus can be made regular by the transversality argument above. Because of the spherical node, this configuration is of codimension at least 2, giving negative expected dimension. A contradiction.

We would also like to rule out disk bubbling in the vertical direction. Suppose u' limits to two disks u_1, u_2 with no boundary markings on u_2 . The energy of u_2 must be positive, hence the maslov index of u_2 is at least 2 by assumption. First we assume that u_1 is non-constant. Then the same configuration containing only u_1 is already regular by choice of perturbation data, thus it has non-negative expected dimension. This shows that this phenomenon is codimension two, which is impossible.

Thus we must have that u_1 is constant.

In the monotone setting, the case $u_1 = \text{constant}$ is usually dealt with at the algebraic level: the different orderings of the edge markings give different signs in d^2 which cancel [26]. However, since we are only considering a single Lagrangian, this can be ruled out by the assumption $\Sigma \geq 2$. Indeed, in any configuration where a (marked or unmarked) vertex between distinct critical points becomes constant and forms a vertical disk bubble in the limit, we can make the linearized operator surjective on the configuration $\Gamma' = \Gamma - \{\text{disk bubble}\}$, which makes Γ' of expected, non-negative dimension. Since $\Sigma \geq 2$, this means that Γ' is a codimension 2 stratum, which contradicts our assumption that $\text{index}(\Gamma) \leq 1$.

In the case when all the input and output critical points are the same, we can rule out disk bubbling by an argument as in [26], which shows that the somewhere injective disks with boundary conditions are of dimension $n + 1$. The image of the evaluation map

$$\begin{aligned} ev : \mathcal{M}(L_{F_p}, J_F, [u_2]) \times_G S^1 &\rightarrow L \\ ev(u, \theta) &= u(\theta) \end{aligned}$$

is then $n + 1 + 1 - 3 = n - 1$ where G is the reparameterization group of D with dimension 3. Thus, the critical points can be avoided in the image with proper choice of perturbation data.

When u_2 is nowhere-injective, we use the covering results of [21] and then apply the previous argument.

□

4.5. The case of a Kähler fiber. When the fibers of our symplectic fibration have a complex structure which is integrable and tames the symplectic form, we can actually achieve transversality by only considering block diagonal complex structures. This is made possible by the *h-principle* of Grauert, which says that for a stein manifold D and a holomorphic lie group G , a continuous map $D \rightarrow G$ can be made holomorphic by a continuous homotopy (see [16, 18]). The main reason for considering this more restrictive case is so that any holomorphic disk into E takes on the form of a pair $(u_B, u_F) : D \rightarrow B \times F$, and the moduli space can be made regular by simply choosing regular perturbation

data for the base and fiber separately.

Pick a perturbation datum J_B which is regular for (B, L_B) . For an isolated, regular J_B -holomorphic disk $u : (D, \partial D) \rightarrow (B, L_B)$ we have the pullback bundle (u^*E, u^*L) . To apply the h-principle, we want view this as a principal bundle. The associated bundle construction is functorial, so let us assume that u^*E is a principal G -bundle for some complex lie group G .

Grauert's h-principle says the following:

Theorem 12. [18] *Let X be a stein space and G a complex lie group. Then any map $f \in C(X, G)$ can be made holomorphic through a homotopy.*

The open disk $D \subset \mathbb{C}$ is a stein space, since it is a domain of holomorphy. Moreover, it is contractible, so any G bundle is topologically trivial, which gives a continuous section $\tilde{\phi} : D \rightarrow u^*E$. The h-principle can be extended to sections of principle bundles, so that means we get a holomorphic section $\phi : D \rightarrow u^*E$ which shows that this bundle is holomorphically trivial. This triviality is preserved by the associated bundle construction, so the net result is that the original bundle is trivial.

Taking this point of view, lifts of holomorphic disks $u : D \rightarrow B$ to E are the same as sections $\tilde{u} : D \rightarrow u^*E$ that satisfy $J_F \circ du = du \circ j$ for some a.c. structure on F . In detail, we consider a.c. structures on E of the form $\begin{bmatrix} J_F & 0 \\ 0 & J_B \end{bmatrix}$ with respect to the connection. By monotonicity of the fiber (see theorem 10), there is a Baire set of taming a.c. structures $\mathcal{J}_{F,\omega}^{reg}$ such that $\mathcal{M}_\Gamma(F, L_F, J_F)$ is smooth of expected dimension for $J_F \in \mathcal{J}_{F,\omega}^{reg}$. If we started off with a coherent system of regular domain-dependent a.c. structures $\{J_{B,\Gamma}\}_\Gamma$ for (B, L_B) , then any $J_F \oplus J_B$ holomorphic disk u is the same as a pair $(\pi \circ u, \widetilde{\pi \circ u}) : D \rightarrow B \times F$ by the triviality of u_B^*E . Thus, $J_F \oplus J_B$ is regular for (E, L) . This gives us the following refinement to theorem 10:

Theorem 13. *If $F \rightarrow E \rightarrow B$ is a fibration with Kähler fibers and compatible symplectic form, then Theorem 10 is achieved by using block diagonal almost complex perturbation data.*

4.6. Homotopy Invariance. The Floer cohomology and A_∞ algebra on the Floer chain complex is expected to be invariant of the choice of coherent perturbation system $(\mathcal{J}_g, f_\Gamma)_\Gamma$ and choice of divisor. The proof of this fact follows almost directly from that of [4] section 3 and the

methods we have used here to achieve transversality and compactness. We summarize the result: For two perturbation systems \mathcal{P}^0 and \mathcal{P}^1 , one develops a theory of *quilted* \mathcal{P}^{01} -holomorphic treed disks, which are \mathcal{P}^0 resp. \mathcal{P}^1 holomorphic at the root resp. leaves, and is \mathcal{P}_t^{01} -holomorphic for some path between \mathcal{P}^0 and \mathcal{P}^1 . The precise statement is:

Theorem 14. [4] *For any stabilizing divisors D^1 and D^2 , and and convergent, coherent, regular, stabilized perturbation systems $\underline{\mathcal{P}}_1$ and $\underline{\mathcal{P}}_2$, the Fukaya algebras $CF(L, \underline{\mathcal{P}}_1)$ and $CF(L, \underline{\mathcal{P}}_2)$ are convergent homotopy equivalent.*

A synopsis of the result is as follows. Pick a time parameterization for each quilted type, which takes 0 on the root, 1 on the leaves, and only depends on the edge distance from the single quilted component. We assume that the two divisors we pick are built from homotopic sections of the same line bundle. Given an energy E , lemma 2 guarantees the existence of a path (or even an open dense set) of a.c structures J_{D^t} such that D_t contains no J_{D^t} -holomorphic spheres. We then take a time dependent perturbation system \mathcal{P}_t^{01} which takes values in the open, dense set guaranteed by lemma 2 and is equal J_{D^t} on the thin part of the domain. Then, transversality and compactness follow for quilted \mathcal{P}_t^{01} treed disks, and we can define a *perturbation morphism* P^{01} from \mathcal{P}^0 to \mathcal{P}^1 on products by taking the isolated \mathcal{P}_t^{01} trajectories. This, in turn defines an A_∞ morphism between the A_∞ algebras $CF(L, \mathcal{P}^0, D^0)$ and $CF(L, \mathcal{P}^1, D^1)$. To show that the composition of the two perturbation morphisms $P^{10} \circ P^{01}$ is homotopic to the identity, one develops a similar theory with *twice-quilted* treed disks.

4.7. Leray-Serre for Floer Cohomology. In the case of a fibered Lagrangian, we would like to compute Floer cohomology with coefficients in some Novikov ring with two variables. Denote

$$\bar{\Lambda}_{\geq 0}^2 := \left\{ \sum_{i,j} c_{i,j} q^{\rho_i} r^{\eta_i} \mid c_{i,j} \in \mathbb{C}, 0 \leq \eta_j, \rho_i \in \mathbb{R}, \eta_j, \rho_i \rightarrow \infty \right\}$$

Choose a brane structure on the Lagrangian L_B (see appendix) and let $\text{Hol}(\pi \circ u)$ be the evaluation of $\pi \circ u$ with respect to the chosen rank one local system $\pi_1(L_B) \rightarrow \Lambda^\times[q]$. Define the A_∞ relation maps as:

$$(21) \quad \mu^n(x_1, \dots, x_n) = \sum_{x_0, [u] \in \overline{\mathcal{M}}_\Gamma(L, D, \underline{x})_0} (-1)^\diamond (\sigma([u])!)^{-1} \text{Hol}_L(\pi \circ u) r^{E([u]) - E([\pi \circ u])} q^{E([\pi \circ u])} \epsilon([u]) < x_0 >$$

Here, $E([\pi \circ u])$ is the energy of $\pi \circ u$ with respect to the form $K\omega_B$. The remaining energy $E([u]) - E([\pi \circ u])$ is the energy of the disk with

respect to the minimal coupling form a . Since $\iota^*a = \omega_F$, this is the fiber energy plus an additional term coming from the connection. Label the critical points in L by x_j^i , where j denotes the y_j such that $\pi(x_j^i) = y_j$. Now let us filter the complex $CF(L, \bar{\Lambda}_{\geq 0}^2)$ by q degree; $\mathcal{F}_q^k CF(L)$ is generated by critical points with coefficients from novikov polynomials of minimal degree $\geq k$ in the q variable.

Assume that L is unobstructed, so there is a solution b to the Maurer-Cartan equation. Let $h_2 : \pi_2(E, L) \rightarrow H_2(E, L)$ be the relative Hurewicz morphism. From the definition of a rational Lagrangian, the image of the energies $[\omega_B] \circ h_2(\pi_2(B, L_B))$ is discrete. This allows us to use a smaller novikov ring:

$$\Lambda_{\geq 0}[q, r] := \left\{ \sum_{i,j} c_{i,j} q^{k_i \rho r^{\eta_j}} \mid c_{i,j} \in \mathbb{C}, 0 \leq k_i \in \mathbb{Z}_{\geq 0}, \eta_j \in \mathbb{R}, k_i, \eta_j \rightarrow \infty \right\}$$

Where ρ is the energy quantization for B . Let us pick a solution b to the Maurer-Cartan equation for the A_∞ algebra $CF(L, \Lambda_{\geq 0}[q, r])$. Then μ_b^1 respects the filtration by q .

Following chapter 6 in [13], we are in the situation called *A Toy Model* (section 6.2). Thus, filtration by q takes us to the following result:

Theorem 15. *Let $F \rightarrow E \rightarrow B$ be a fibration of symplectic manifolds with the weak coupling form, along which we have a fibration of Lagrangians $L_F \rightarrow L \rightarrow L_B$, and a divisor $D = \pi^{-1}(D_B)$ for a stabilizing divisor D_B of large enough degree in the base. Choose a regular, coherent, stabilizing, convergent perturbation datum (\mathcal{P}_Γ) . Then there is a spectral sequence $E_s^{p,q}$ which converges to $HF^*(L, \Lambda[r, q])$ whose second page is the Floer cohomology of the family of L_F over L_B . The latter is computed by a spectral sequence with second page*

$$(22) \quad \tilde{E}_2^* = H^*(L_B, \mathcal{HF}(L_F, \Lambda_{\geq 0}[r])) \otimes gr(\mathcal{F}_q \Lambda_{\geq 0}[q])$$

where the coefficients come from the system which assigns the module $HF(L_{F_p}, \Lambda_{\geq 0}[r])$ to each critical fiber.

Proof. This is due to the observations in [13] chapter 6 and our construction of the Morse-Floer chain complex. Since the possible energies for L form a discrete subgroup, the differential $\mu_b^1 := \delta$ is *gapped* with respect to the filtration for any solution b to the Maurer-Cartan equation. Thus, by similar arguments as in [13] section 6.3, the spectral sequence corresponding to this filtration converges to $H(CF(L), \mu_b^1)$.

It remains to calculate the second page. Let

$$(23) \quad Z_s^q = \{x \in \mathcal{F}^q CF(L) \mid \mu_b^1(x) \in \mathcal{F}^{q+s-1} CF(L)\} + \mathcal{F}^{q+1} CF(L)$$

$$(24) \quad B_s^q = \{\mu_b^1(\mathcal{F}^{q-s+2} CF(L)) \cap \mathcal{F}^q CF(L)\} + \mathcal{F}^{q+1} CF(L)$$

$$(25) \quad E_s^q = Z_s^q / B_s^q$$

By definition, we have

$$E_1^* \cong CF(L) \otimes_{\mathbb{C}} gr_*(\mathcal{F}\Lambda_{\geq 0}[q])$$

where

$$gr_*(\mathcal{F}\Lambda_{\geq 0}[q]) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}[q]} gr_n(\mathcal{F}\Lambda_{\geq 0})$$

is the associated graded module of $\Lambda_{\geq 0}[q]$, and the differential on E_1^* is induced from the Floer differential on $CF(L)$. Let us decompose this differential by taking the projections onto the degree n part: let $\delta_n = \pi_n \circ \delta$. Then

$$\delta(x) = \sum_{n \geq 0} \delta_n(x) q^{n\rho}$$

where ρ is the energy quantization constant for $(B, K\omega_B)$. Taking $s = 2$ we see that E_2^* is then $\text{Ker}(\delta_0)/\text{Im}(\delta_0)$ which is the Floer cohomology of the *family* of L_F over L_B . In other words, it is the cohomology of the complex $C(L, \Lambda_{\geq 0}[q, r])$ with δ_0 , which is the zero q -degree part of the differential δ . The usual Leray-Serre theorem for fiber bundles tells us that this complex has cohomology which can be calculated via a spectral sequence whose second page is the cohomology of the base with local coefficient system as the assignment of the modules $HF(L_{F_p}, \Lambda_{\geq 0}[r])$ to each critical fiber.

□

5. APPLICATIONS

5.1. Full flag manifolds. The procedure for finding Floer-non-trivial Lagrangians in $\text{Flag}(\mathbb{C}^3)$ can be generalized to full flags in higher dimensional complex vector spaces. We give a straight-forward procedure.

Let V_i be an i -dimensional subspace of \mathbb{C}^n , and consider the map $\pi : \text{Flag}(\mathbb{C}^n) \rightarrow \mathbb{P}^n$ given by

$$\pi(V_1 \subset \cdots \subset V_{n-1}) = V_1$$

The fiber of this map is naturally realized as $\text{Flag}(\mathbb{C}^{n-1})$ consisting of chains of subspaces

$$V_2/V_1 \subset \cdots \subset V_{n-1}/V_1 \subset \mathbb{C}^n/V_1$$

In addition, the fibers are holomorphic in the standard structure, and so they are Kähler. The KKS form determines a symplectic connection $H = TF^{\omega_{KKS}^\perp}$, from which we can construct a minimal coupling form ω_H which is fiber-wise equal to ω_{KKS} , with associated weak coupling form $\omega_H + K\pi^*\omega_{\mathbb{P}^n}$ (details given in section 2). Let us assume that there is a non-displaceable Lagrangian $L_{n-1} \subset \text{Flag}(\mathbb{C}^{n-1})$. $\text{Flag}(\mathbb{C}^n)$ is simply connected, so by theorem 2 we get a new connection H' , weak form $\omega_{H'} + K\pi^*\omega_{\mathbb{P}^n}$, and a symplectic isotopy f_t such that

$$f_1^*(\omega_{H'} + K\pi^*\omega_{\mathbb{P}^n}) = \omega_H + K\pi^*\omega_{\mathbb{P}^n}$$

such that the bundle is $\omega_{H'} + K\pi^*\omega_{\mathbb{P}^n}$ trivial above $\text{Cliff}(\mathbb{P}^n)$. For simplicity, let us take the product Lagrangian

$$L_{n-1} \times \text{Cliff}(\mathbb{P}^n)$$

Denote by L_n the image of this product under f_1^{-1} .

It is not unreasonable to expect that the Floer cohomology of L_n is non-trivial. In fact, since we chose the (topological) product lagrangian, we can choose our Morse-Smale function so that the action of $\pi_1(\text{Cliff}(\mathbb{P}^n))$ on the groups $HF(L_F, \Lambda_{\geq 0}[r])$ is trivial. Thus the second page the the spectral sequence 22 is the tensor product

$$E_2^* = H(L_B) \otimes HF(L_F, \Lambda_{\geq 0}[r]) \otimes gr(\mathcal{F}_q \Lambda_{\geq 0}[q])$$

with differential which counts configurations with maslov index two in the base, i.e.

$$\begin{aligned} \delta_2 : H(L_B) \otimes HF(L_F, \Lambda_{\geq 0}[r]) \otimes \mathcal{F}_q^n \Lambda_{\geq 0}[q] \\ \rightarrow H(L_B) \otimes HF(L_F, \Lambda_{\geq 0}[r]) \otimes \mathcal{F}_q^{n+2} \Lambda_{\geq 0}[q] \end{aligned}$$

For the higher pages, we have a similar expression but with the $n+2$ in the filtration replaced with $n+2k$ with $k = s-1$ and s as the page number. Moreover, since the fibration of lagrangians is trivial, we have a well defined projection $\pi_{\text{vert}} : L_n \rightarrow L_{n-1}$ which preserves the energy of vertical disks. Because of this, any relations induced by the higher order differentials involving the variable r are redundant, due to the fact that we have already considered these with the differentials δ_0 and δ_1 . Thus, the resulting spectral sequence only considers the quantum contributions in the variable q , and we arrive at the formula

$$HF(L, \Lambda_{\geq 0}[q, r]) \cong HF(L_B, \Lambda_{\geq 0}[q]) \otimes HF(L_F, \Lambda_{\geq 0}[r])$$

5.2. Projective ruled surfaces. There are some low dimension applications, which naturally show up in the Gonzalez-Woodward symplectic minimal model program [33, 15]. In dimension 4, a typical end stage of *running* of the minimal model program is a so called *ruled surface*, or a holomorphic \mathbb{P}^1 bundle over a Riemann surface. These occur in the classification of surfaces due to Enriques-Kodaira [2], which we review in this section. Then, we show that one can construct a fibered Lagrangian torus which is Floer-non-trivial.

In the classification of projective surfaces [2], there is the case where no exterior powers of the canonical line bundle admit holomorphic sections. More precisely, let X be a projective surface, and $K_X = TX \wedge TX$ be the *canonical line bundle*. We have the object $H^0(X, K_X)$, whose dimension counts the number of non-vanishing holomorphic (or algebraic) sections up to scaling by functions in the structure sheaf. Form the sequence of integers $P_i(X) = \dim H^0(X, K_X^{\otimes i})$. If $P_i(X) = 0$ for all positive integers i , then the *Kodaira dimension* of X is said to be $-\infty$ (This is in contrast to the other possible cases when $P_i(X)$ has asymptotics like i^k for $k \geq 0$). These are the so-called *ruled surfaces* where X fibers as a \mathbb{P}^1 bundle over a Riemann surface C . For the complete classification, see [2].

Basic cohomology theory gives us that any complex analytic \mathbb{P}^n bundle with structure group $PGL(n+1, \mathbb{C})$ over a Riemann surface C is actually the projectivization of a vector bundle. This follows from the long exact sequence of sheaf cohomology groups arising from the sequence

$$(26) \quad 0 \rightarrow \mathcal{O}_C^* \rightarrow GL(n+1, \mathbb{C}) \rightarrow PGL(n+1, \mathbb{C}) \rightarrow 0$$

and the appropriate GAGA result that says every analytic vector bundle over B is algebraic [2]. Thus, we restrict our attention to rank 2 algebraic vector bundles over Riemann surfaces.

5.2.1. Example: Base curve \mathbb{P}^1 . Restricting further to rank 2 bundles over \mathbb{P}^1 , a theorem of Grothendieck tells us that every such vector bundle splits as $\mathcal{O}(k) \oplus \mathcal{O}(l)$. Since $\mathbb{P}(V) \cong \mathbb{P}(V \otimes \mathcal{O}(n))$, we can normalize the description of the fiber bundle as $\Sigma_n := \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n))$ for some $n \geq 0$. These are known as the Hirzebruch surfaces. For each n take a kähler form which is compatible with the standard complex

structure. The fibers of π are then holomorphic spheres, and thus symplectic.

For the sphere, we know that any embedded loop which divides the total symplectic area in half is a non-displacable, monotone Lagrangian. More simply, we want to see that given a two equators that a) we can generate a fibered Lagrangian in the total space Σ_n and b) that this Lagrangian is Floer-non-trivial.

To actually generate a Lagrangian, we deform the connection and use parallel transport to flow out a torus. Let L_B be an equator in the base S^2 with parameterization γ_B , and let ω define a connection on Σ_n by $TF \oplus H$ with $H = TF^{\omega\perp}$. Then, parallel transport along γ_B gives maps

$$\phi_s : \pi^{-1}(\gamma(0)) \rightarrow \pi^{-1}(\gamma(s))$$

which are Hamiltonian diffeomorphisms since S^2 is simply connected. Following Seidel [28] section 15, we then deform the symplectic form (and horizontal splitting) by $d\alpha$ where α vanishes on TF to prescribe different parallel transport maps. This is made precise in the following lemma:

Lemma 5. [28] *Let γ be a path in B and let ψ_s be a Hamiltonian isotopy of $F_{\gamma(0)}$ starting with $\psi_0 = Id$. Then there is a deformation of the fibration along γ which extends to all of E such that the parallel transport maps along γ satisfy*

$$\phi = \phi_s \circ \psi_s$$

In particular, when γ is a loop as in this case, we can deform the structure so that parallel transport around the loop is the identity. Thus, pick a simple closed curve $L_F \subset F$ and look at the image of its transport along γ . This gives us a Lagrangian torus in the deformed manifold.

Proof of lemma. The idea is as follows: Let α be a 1-form which vanishes on TF and in a neighborhood of $\pi^{-1}(\gamma)$. Then $\omega + d\alpha = \omega$ when restricted to TF , and so is non-degenerate. Let $Y^\sharp = (X, Y)$ (in the ω splitting) be a horizontal lift (in the $\omega + d\alpha$ splitting) of a vector field Y . Then, $\mathcal{L}_{Y^\sharp}\alpha = 0$ on TF since it is the pullback of a base form, and vertically we have

$$\begin{aligned}
0 &= \iota_{Y\#}(\omega + d\alpha) \\
&= \iota_X\omega + \iota_{Y\#}d\alpha \\
&= \iota_X\omega - d\iota_Y\alpha
\end{aligned}$$

Which says that parallel transport in the $\omega + d\alpha$ splitting is infinitesimally the Hamiltonian flow of $-\iota_Y\alpha$. Thus, for a Hamiltonian isotopy ψ_s with associated time-dependent Hamiltonian H_s let α be any 1-form which vanishes on TF such that

$$\iota_Y\alpha|_{\pi^{-1}(\gamma(s))} = H_s$$

and vanishes outside of a neighborhood of $\pi^{-1}(\gamma)$. Parallel transport with respect to the α splitting will then be prescribed by $\phi_s \circ \psi_s$. The desired deformation is then

$$\omega + K\pi^*\omega_B \mapsto \omega + td\alpha + K\pi^*\omega_B$$

for K large enough. □

The deformation only changes the symplectic form by an exact form. Thus, an application of Moser's theorem gives us a symplectic isotopy back to the original symplectic structure, which in turn gives a Lagrangian.

To show Floer non-triviality, we can compute the homology directly. Since the fibers are Kähler, any J holomorphic disk $u : D \rightarrow \Sigma_n$ is holomorphically a product of two disks $\psi \times \phi : D \rightarrow S^2 \times S^2$. Since L is a torus (we assume L is orientable), we can use the usual Morse-Smale function $(s, t) \mapsto h(s) + h(t)$ with h being the height function on S^1 , and s resp. t the base resp. fiber coordinate. J -holomorphic disks in the base and fibers are hemispheres and multiple covers of such. In this case, we can directly compute the differential on the Morse complex and thus the Floer cohomology. Let us denote the critical points: x_0 as the maximum, x_1 as the minimum in s and maximum in t , x_2 the minimum in t and maximum in s , and x_3 the global minimum.

Using the index formula

$$0 = \text{Ind}(x_{out}) - \text{Ind}(x_{in}) + \sum_{i=1}^m I_i(u) - 1$$

one finds that:

$$\begin{aligned} d_q(x_0) &= 0; \\ d_q(x_1) &= q^2x_0 - q^2x_0 = 0; \\ d_q(x_2) &= r^2x_0 - r^2x_0 = 0; \\ d_q(x_3) &= r^2x_1 - r^2x_1 + q^2x_2 - q^2x_2 = 0; \end{aligned}$$

Thus the Floer cohomology is isomorphic (as a Λ -module) to the Morse cohomology with Novikov coefficients.

5.2.2. Base Curve with genus ≥ 2 . Let B be a Riemann surface of genus ≥ 2 . Considering Lagrangians as simple closed curves, it has been observed by Seidel, Efimov and others [10, 29] that the generators of the Fukaya category are given by *balanced* curves, which are nullhomologous curves that satisfy

$$(27) \quad \frac{\text{Area}(B_+)}{\chi(B_+)} = \frac{\text{Area}(B_-)}{\chi(B_-)}$$

whenever L_B divides B into two Riemann surfaces with boundary (in particular, L_B is not a contractible curve when $g \geq 2$). This is really a monotonicity condition of sorts, which allows one to construct the Lagrangian intersection theory (see [29, 10]).

The standard long exact sequence in homology gives us

$$0 \rightarrow H_2(B) \rightarrow H_2(B, L_B) \rightarrow H_1(L_B) \rightarrow H_1(B)$$

The last map is injective since L_B is not a contractible curve, thus we get that $H_2(B) \cong H_2(B, L_B)$. From a similar exact sequence involving homotopy groups, we see that there are no non-trivial disks with boundary in L_B . Thus, the Morse-Floer homology groups are isomorphic to the classical Morse homology groups, which shows this type of L_B as non-displaceable.

Now let V be a rank 2 vector bundle over B and $\mathbb{P}(V) \rightarrow E \rightarrow B$ be its projectivization. Let us pick a Lagrangian with $L_F \subset \mathbb{P}^1$ dividing the symplectic area of the sphere into two equal parts. As above, we have that $HF(L_F, \Lambda_{\geq 0}) \cong H^{\text{Morse}}(L_F, \Lambda_{\geq 0})$.

Finding a sub-bundle $L_F \rightarrow L \rightarrow L_B$ amounts to the same type of question that was answered in the previous example. Thus, let us

assume that we can deform the connection in a neighborhood of L_B so that we may choose a consistent section of balanced Lagrangians L_{F_p} giving rise to a fibered Lagrangian L .

This situation is nice enough that we can use the spectral sequence to compute $HF(L, \Lambda_{\geq 0}[q, r])$. According to our main result, the second page is the cohomology of the Morse chain complex of L_B with coefficients in the local system $\mathcal{HF}(L_F, \Lambda_{\geq 0}[q, r])$. The filtration is with respect to the base energy, but the differential induced on any of the higher pages does not include any q terms. Therefore, the sequence collapses after the second page, and we have that the Floer cohomology of L is isomorphic to the homology of the complex $CF(L)$ with differential δ_0 which counts isolated Floer trajectories in each fiber in addition to zero-energy Morse configurations in the base:

$$gr_*(HF(L, \Lambda_{\geq 0}[q, r])) \cong E_2(CF(L), \Lambda_{\geq 0}[q, r], \delta_0, \mathcal{F}_q)$$

According to our main theorem, the second page of the Floer fibration spectral sequence can be computed via the usual Leray-Serre spectral sequence of a fibration with vertical differential given by δ_F^{Floer} and horizontal differential given by δ_B^{Morse}

$$E_2(CF(L, \Lambda_{\geq 0}[q, r]), \delta_0, \mathcal{F}_q) \cong E_{\infty}^{LS}(C(L), \delta_B^{\text{Morse}} \pm \delta_F^{\text{Floer}}, \mathcal{F}_d)$$

where the filtration \mathcal{F}_{bd} is given by *base degree*, i.e. $\deg \pi(x_i)$ for x_i a critical point on the total space. The second page of this is given as follows

$$E_2^{LS} \cong gr_*(H(L_B, \mathcal{HF}(L_F, \Lambda_{\geq 0}[r])))$$

6. APPENDIX

For completeness, we include some aspects of the A_{∞} -algebra and Maurer-Cartan equation for a rational Lagrangian in a rational symplectic manifold. This section is taken from [4].

6.1. A_{∞} algebras and composition maps. We define the necessary algebraic notions to consider Fukaya algebras of Lagrangians. Define the *universal Novikov field* of formal power series:

$$(28) \quad \Lambda = \left\{ \sum_i c_i q^{\rho_i} \mid c_i \in \mathbb{C}, \rho_i \in \mathbb{R}, \rho_i \rightarrow \infty \right\}$$

The subalgebra of only non-negative powers will be denoted $\Lambda_{\geq 0}$ (similarly $\Lambda_{> 0}$).

The axioms for an A_∞ algebra are as follows. Let A be a \mathbb{Z}_g -graded vector space and let

$$\mu^d : A^{\otimes d} \rightarrow A[2-d]$$

be multilinear maps. (A, μ^d) is said to be an A_∞ algebra if the composition maps satisfy the following relations:

$$0 = \sum_{n,m \geq 0} \sum_{n+m \leq d} (-1)^{n+\sum_{i=1}^n |a_i|} \mu^{d-m+1}(a_1, \dots, a_n, \mu^m(a_{n+1}, \dots, a_{n+m}), a_{n+m+1}, \dots, a_d)$$

We will also need the notion of an A_∞ morphism between two algebras. Let A_0 and A_1 be two A_∞ algebras.

Definition 27. An A_∞ morphism from A_0 to A_1 is a collection of maps

$$\mathcal{F}^d : A_0^{\otimes d} \rightarrow A_1[1-d], \quad d \geq 0$$

such that the following equation holds:

$$\sum_{i+j \leq d} (-1)^{i+\sum_{j=1}^i |a_j|} \mathcal{F}^{d-j+1}(a_1, \dots, a_i, \mu_{A_0}^j(a_{i+1}, \dots, a_{i+j}), a_{i+j+1}, \dots, a_d) = \sum_{i_1+\dots+i_m=d} \mu_{A_1}^m(\mathcal{F}^{i_1}(a_1, \dots, a_{i_1}), \dots, \mathcal{F}^{i_m}(a_{i_1+\dots+i_{m-1}+1}, \dots, a_d))$$

In order to properly define the Fukaya algebra for a Lagrangian, we require that the Lagrangian have additional structure, called a *brane structure*. Let E be a symplectic manifold and $\text{Lag}(E)$ the fiber bundle whose fiber at p is the grassmanian of Lagrangian subspaces of $T_p E$. For an even integer g , a *Maslov cover* is a g -fold cover $\text{Lag}^g(E) \rightarrow \text{Lag}(E)$ such that the induced two-fold cover $\text{Lag}^g(E)/\mathbb{Z}_{g/2} \rightarrow \text{Lag}(E)$ is the oriented double cover. A Lagrangian submanifold is *admissible* if it is compact and oriented (we assume connectedness for now).

A *grading* on L is a lift of the canonical map

$$L \rightarrow \text{Lag}(X), \quad l \mapsto T_l L$$

to $\text{Lag}^g(X)$. A *relative spin structure* for L is a lift of the transition maps $\psi_{\alpha\beta}$ for TL to Spin satisfying the cocycle condition

$$\psi_{\alpha\beta} \psi_{\alpha\gamma}^{-1} \psi_{\beta\gamma} = i^* \varepsilon_{\alpha\beta\gamma}$$

where $\varepsilon_{\alpha\beta\gamma}$ is a 2-cycle on E . Let

$$\Lambda^\times = \{c_0 + \sum_{i>0} c_i q^{p_i} \in \Lambda_{\geq 0} \mid c_0 \neq 0\}$$

be the subgroup of formal power series with invertible leading coefficient. A *rank one local system* (with values in Λ^\times) is a representation

$\pi_1(E) \rightarrow \Lambda^\times$. A *brane structure* for a compact oriented (connected) Lagrangian L consists of the following data:

- (1) A Maslov cover $\text{Lag}^g(E) \rightarrow$ with a grading,
- (2) A rank one local system with values in Λ^\times and
- (3) A relative spin structure with the given 2-cycles.

An *admissible Lagrangian brane* is an admissible Lagrangian submanifold equipped with a brane structure. For such an object, the space of Floer cochains is defined as

$$CF(L) = \bigoplus_{d \in \mathbb{Z}_g} CF_d(L), \quad CF_d(L) = \bigoplus_{x \in \hat{\mathcal{L}}_d(L)} \Lambda \langle x \rangle$$

Given a Lagrangian brane L , we denote by $\text{Hol}_L(u) \in \mathbb{C}^\times$ the evaluation of the local system on the homotopy class of loops defined by going around the boundary of the treed disk once. We denote by $\sigma([u])$ the number of interior markings of $[u] \in \overline{\mathcal{M}}_\Gamma(L, D, \underline{x})$.

Definition 28. [4] For regular stabilizing coherent perturbation data (P_Γ) define the composition maps

$$\mu^n : CF(L)^{\otimes n} \rightarrow CF(L)$$

on critical points by the following equation:

$$(29) \quad \mu^n(x_1, \dots, x_n) = \sum_{x_0, [u] \in \overline{\mathcal{M}}_\Gamma(L, D, \underline{x})_0} (-1)^\diamond (\sigma([u])!)^{-1} \text{Hol}_L(u) q^{E([u])} \epsilon([u]) x_0$$

where $\diamond = \sum_{i=1}^n i|x_i|$.

So far, we have neglected to mention anything about units. In fact, everything that has been recorded so far can be done to incorporate a *strict unit*.

Definition 29. [4] Let A be an A_∞ algebra. A *strict unit* for A is an element e_A such that

$$\begin{aligned} \mu^2(e_A, a) &= a = (-1)^{|a|} \mu^2(a, e_A) \\ \mu^n(\dots, e_A, \dots) &= 0, \quad n \neq 2 \end{aligned}$$

An A_∞ -algebra is called *strictly unital* if it is equipped with a strict unit.

One obtains such a thing by replacing the unique maximum x with 3 copies such that

$$i(x_M^\bullet) = i(x_M^\circ) = 0, \quad i(x_M^\Delta) = -1$$

The notion of a treed holomorphic disk, morphisms of moduli spaces, and a coherent perturbation system can be modified to incorporate these three additional copies. See [4] for the full details.

Let $\widehat{CF}(L)$ be the chain complex with this additional structure
We have the following theorem.

Theorem 16 (A_∞ relations). [4] *Let \mathcal{P} be a coherent, stabilizing, regular perturbation datum. Then $(\widehat{CF}(L), \{\mu^n\}_n)$ is A_∞ algebra with strict unit. The subcomplex $CF(L)$ is an A_∞ -algebra without unit.*

Sketch of proof. For an admissible tuple (x_0, \dots, x_n) , components of the moduli space $\overline{\mathcal{M}}(L, D, \underline{x})_1$ are compact manifolds with (possibly overlapping) boundary. Thus they obey the following relation:

$$(30) \quad 0 = \sum_{\Gamma \in \mathfrak{M}_{n,m}} \sum_{[u] \in \partial \overline{\mathcal{M}}_\Gamma(L, D, \underline{x})_1} (\sigma(u))^{-1} \varepsilon(u) q^{E(u)} \text{Hol}(u)$$

When Γ is a type without weights, then the boundary points of the moduli space are types with a (one additional) breaking, equivalent to the union of types Γ_1 and Γ_2 with n_1 resp. $n - n_1 - 1$ leaves. By the (product axiom),

$$(31) \quad \partial \overline{\mathcal{M}}(L, D, \underline{x})_1 = \bigcup_{y, \Gamma_1, \Gamma_2} \mathcal{M}_{\Gamma_1}(L, D, x_0, \dots, x_{i-1}, y, x_{i+n_2}, \dots, x_n) \times \mathcal{M}_{\Gamma_2}(L, D, y, x_i, \dots, x_{i+n_2-1})$$

Say $\sigma([u]) = m$ Since there are m choose m_1, m_2 ways of distributing the interior markings to the two component graphs,

$$(32) \quad 0 = \sum_{\substack{i, m_1+m_2=m \\ [u_1] \in \mathcal{M}_{\Gamma_1}(L, D, x_0, \dots, x_{i-1}, y, x_{i+n_2}, \dots, x_n)_0 \\ [u_2] \in \mathcal{M}_{\Gamma_2}(L, D, y, x_i, \dots, x_{i+n_2-1})_0}} (m!)^{(-1)} \binom{m}{m_1} q^{E(u_1)+E(u_2)} \varepsilon(u_1) \varepsilon(u_2) \text{Hol}_L(u_1) \text{Hol}(u_2)$$

This is the A_∞ relation up to signs, and it now remains to show that the signs arising from the orientations are consistent with those of the A_∞ relations. We refer the reader to [4].

□

Next, we include the necessary statements to find a perturbation system so that the resulting A_∞ algebra is convergent:

Definition 30. [4] A perturbation system $\underline{P} = (P_\Gamma)$ is *convergent* if for each energy bound E , there exists a constant $C(E)$ such that for any Γ and any treed J_Γ -holomorphic disk $u : C \rightarrow X$ of type Γ , the total Maslov index $I(u) := \sum I(u_i)$ satisfies

$$(33) \quad (E(u) < E) \Rightarrow (I(u) < c(E)).$$

Lemma 6. [4] Any convergent, coherent, regular, stabilizing perturbation system $\underline{P} = (P_\Gamma)$ defines a convergent Fukaya algebra $\widehat{CF}(L, \underline{P})$.

Proposition 1. [4] There exist convergent, coherent, regular, stabilizing perturbations $\underline{P} = (P_\Gamma)$.

See [4] for the proof.

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